FORMALITY OF ABELIAN ARRANGEMENTS

A CLASSIFICATION ANSWERING PAGARIA'S QUESTION 4.10, WITH THE FORMALITY PART OF SUCIU'S PROBLEM 8.3

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Abstract. Let T be a complex abelian variety and let X be the complement of a finite collection of translates of connected codimension-one abelian subvarieties. We prove that X is rationally formal precisely when the distinct holomorphic normal lines are independent. Translations, parallel divisors, and disconnected intersections are all allowed. The obstruction is a simultaneous choice of primitives: formality must respect every incidence relation in one Sullivan map, and a circuit among the normal lines violates that requirement. The same argument shows that a circuit of length c obstructs $(c-2)$-formality. Independent normals give a finite isogeny cover by an abelian variety times punctured elliptic curves. For the entire formal class we determine the degree-one generated cohomology subalgebra, the rational holonomy Lie algebra, and every LCS and Chen rank. Within this class, connectedness of all nonempty intersections is equivalent to degree-one generation of full cohomology, Koszulness, the standard Poincaré–LCS identity, and the rational $K(\pi,1)$ property. Every formal complement is an ordinary $K(\pi,1)$, even when these additional equivalent conditions fail. The elliptic specialization answers Pagaria's Question 4.10 in its original, possibly non-unimodular range.

1. The question and the classification

An elliptic arrangement in E^n consists of connected fibers of integer characters $E^n \to E$, with arbitrary translations. Pagaria's Question 4.10 asks which complements are formal [8]. Theorem 4.8 of that work determines 1-formality for graphical elliptic arrangements, but full formality requires a different obstruction. Suciu's Problem 8.3 also asks about formality, alongside resonance [9]. We answer the formality question; we have not yet resolved its separate resonance part.

The natural geometric setting is a complex abelian variety T. Throughout, an *abelian arrangement* means a finite collection of distinct divisors

$$D_j = t_j + H_j \subset T,$$

where H_j is a connected codimension-one abelian subvariety. This definition does not include arbitrary divisors or higher-codimension subvarieties. Write $n = \dim_{\mathbb{C}} T$ and

$$X = T \setminus \bigcup_j D_j.$$

Divisors are parallel when their direction subgroups agree. Let $I_1, \dots, I_s$ be the parallel classes, with direction subgroups H_λ. The quotient

$$\pi_\lambda : T \longrightarrow E_\lambda = T/H_\lambda$$

is an elliptic curve quotient. Its holomorphic normal line is

$$L_\lambda = \pi_\lambda^* H^{1,0}(E_\lambda) \subset H^{1,0}(T).$$

Let $S_\lambda \subset E_\lambda$ be the finite set whose fibers are removed in class I_λ, and let $m_\lambda = |S_\lambda|$.

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Theorem 1.1. *For an abelian arrangement as above, the following conditions are equivalent:*

(1) *X is formal over $\mathbb{Q}$;*
(2) *the sum $L_1 + \cdots + L_s$ is direct;*
(3) *the number s of parallel classes equals their normal rank.*

When these conditions hold, there is a finite Galois cover of X of the form

$$K \times \prod_{\lambda=1}^{s} (E_\lambda \setminus T_\lambda), \tag{1}$$

where K is an abelian variety of dimension $n - s$ and each T_λ is finite and nonempty. The deck transformations are restrictions of translations. In particular, every such X is an ordinary $K(\pi, 1)$.

No unimodularity, centrality, or connectedness of higher intersections is assumed. The criterion is combinatorial: the layers poset records the parallel classes and the normal rank. This is an explicit classification; the combinatorial determination of formality for elliptic arrangements was already established by Pagaria [8].

All spaces considered here are connected and have finite CW homotopy type. Formality means Sullivan formality of the entire rational cdga, rather than only 1-formality of the fundamental group. We use scalar extension to $\mathbb{C}$ for the obstruction. For finite-type cdgas in characteristic zero, formality is unchanged under extension of the ground field; see [10, §10.1].

2. One map must realize all choices of primitives

The obstruction uses free closed expressions before imposing the arrangement relations. The following lemma explains why their images must be compatible.

Lemma 2.1 (Simultaneous primitives)**.** *Let A be a nonnegative cdga over a characteristic-zero field k, with $A^0 = k$. Let $B = (\Lambda W, d)$ be a Sullivan algebra all of whose free generators have degree one, and let $f : B \to A$ be a cdga map. If A is formal, there is a cdga map*

$$\tau : B \longrightarrow (H^*(A), 0), \qquad H(\tau) = H(f).$$

Proof. Choose a minimal model $\mu : P \to A$. Formality gives a quasi-isomorphism $\nu : P \to (H^*(A), 0)$. Postcomposing ν with the algebra automorphism $H(\mu)H(\nu)^{-1}$ makes $H(\nu) = H(\mu)$.

We construct $g : B \to P$ with $\mu g = f$ by the Sullivan ordering. Suppose all generators preceding w have been lifted. The element $g(dw) \in P^2$ is closed and its image is $df(w)$, so injectivity of $H^2(\mu)$ gives $z \in P^1$ with $dz = g(dw)$. The difference $\mu(z) - f(w)$ is closed. Surjectivity of $H^1(\mu)$ gives a closed $c \in P^1$ whose image represents this difference. Since $dA^0 = 0$, two cohomologous closed degree-one elements of A are equal. Thus $g(w) = z - c$ is a strict lift. Closed generators are lifted using the same H^1 surjectivity. At a limit stage every expression uses only finitely many generators, so the construction defines g on all of B. Set $\tau = \nu g$. □

We will also need a finite-degree version. A q-equivalence induces cohomology isomorphisms through degree q and an injection in degree $q + 1$. Here q-formality is defined by such equivalences, as in [4, Definitions 2.1 and 2.4]; it is not the different convention sometimes called partial formality.

Lemma 2.2. *Suppose $q \geq 1$, $A^0 = k$, and $H^{q+1}(A)$ is finite dimensional. In the setting of Lemma 2.1, if A is q-formal, there is a map*

$$\tau : B \longrightarrow H_{\leq q+1}(A) := H^*(A)/H^{>q+1}(A)$$

which induces $H^j(f)$ in every degree $j \leq q + 1$.

Proof. The q-minimal-model characterization supplies maps $\mu : P \to A$ and $\nu : P \to (H^*(A), 0)$ which are q-equivalences. Put $u = H(\mu)$ and $v = H(\nu)$. For $j \leq q$ set $\sigma_j = u_j v_j^{-1}$. These maps respect multiplication whenever the total degree is at most q.

In degree $q+1$, set $U = v_{q+1}H^{q+1}(P)$ and $U' = u_{q+1}H^{q+1}(P)$. The maps in this degree are injective, so $\rho = u_{q+1}v_{q+1}^{-1} : U \to U'$ is an isomorphism. Every positive-degree decomposable element of degree $q+1$ belongs to both U and U': its factors have degrees at most q and can be lifted through either u or v. Moreover, for $a \in H^i(A)$ and $b \in H^{q+1-i}(A)$,

$$\rho(ab) = \sigma_i(a)\sigma_{q+1-i}(b).$$

Choose complements to U and U' and extend ρ to an automorphism σ_{q+1} of $H^{q+1}(A)$. The finite-dimensional hypothesis ensures that the complements have equal dimension. Together with $\sigma_0 = 1$, these maps define an automorphism σ of the truncated algebra $H_{\le q+1}(A)$: multiplication below the top degree was checked already, multiplication in the top degree was just checked on all decomposables, and larger products are zero in the truncation.

After replacing ν by its truncated composite with σ, its cohomology map agrees with μ through degree $q+1$. The strict lifting argument of Lemma 2.1 still applies, since $H^1(\mu)$ is surjective and $H^2(\mu)$ is injective even when $q = 1$. Composing this lift with the normalized truncated map proves the assertion. We do not extend σ to an automorphism of the untruncated cohomology algebra. □

3. The full layers model and the incidence constraint

We use Dupont's Orlik–Solomon model for a hypersurface arrangement in a smooth projective variety [3, §§2.4,4.4 and Theorem 5.6]. Every connected component of every nonempty intersection is a separate layer. If F has codimension c, the corresponding summands have the form

$$H^p(F;\mathbb{C}) \otimes \mathrm{OS}_F, \qquad \text{total degree } p+c, \tag{2}$$

where OS_F is the local Orlik–Solomon piece of degree c. The product uses restriction maps and the local Orlik–Solomon product; the differential uses Gysin maps and local boundary maps. This cdga, denoted A, has the minimal model of X. We retain all layers. In particular, we do not assume a degree-one presentation in the non-unimodular case.

Every layer is a translate of an abelian subvariety. Give a summand $H^{u,v}(F) \otimes \mathrm{OS}_F$ the auxiliary type $(u+c, v+c)$. Restrictions preserve Hodge types, and the $(1,1)$ shift of a Gysin map cancels the decrease of c by one. Both multiplication and differential preserve this auxiliary bigrading. Since $u+v=p$,

$$A^k_{a,b} = 0 \quad \text{if } a > k \text{ or } b > k. \tag{3}$$

In particular, the type $(r,1)$ part of degree r has exactly one residue:

$$A^r_{r,1} = \bigoplus_j H^{r-1,0}(D_j)e_j, \qquad A^{r-1}_{r,1} = 0. \tag{4}$$

Indeed $v+c=1$ allows $c=0$ or 1, and $c=0$ would give total degree $u+v=r+1$, not r. Thus $c=1$, $v=0$, and $u=r-1$.

Choose a nonzero $\alpha_\lambda \in L_\lambda$. The divisor class of every member of I_λ is a nonzero scalar multiple of $\alpha_\lambda\overline{\alpha}_\lambda$. Rescale all its local generators by that same scalar, so that for $j \in I_\lambda$,

$$de_j = \alpha_\lambda\overline{\alpha}_\lambda, \qquad \alpha_\lambda e_j = \overline{\alpha}_\lambda e_j = 0. \tag{5}$$

The multiplication identities express restriction to the divisor, where the normal classes vanish. The normalizations are over $\mathbb{C}$ and need not be real. The class is nonzero because it is the pullback of a nonzero point class under an elliptic quotient.

There are no degree-one boundaries, and the model gives

$$H^1(A) = H^{1,0}(T) \oplus H^{0,1}(T) \oplus \mathcal{R}, \qquad \mathcal{R} = \left\{ \sum_j c_j e_j : \sum_j c_j de_j = 0 \right\}. \tag{6}$$

The three summands have types $(1,0),(0,1),(1,1)$. At this point $\mathcal{R}$ may contain relations across distinct directions.

Proposition 3.1 (Incidence restricts each primitive)**.** *Let B be the free exterior algebra on a basis of $H^1(T;\mathbb{C})$ and symbols E_j, with closed ambient generators and $dE_j = de_j$. Let $f : B \to A$ send the symbols to the actual model elements. Suppose a map τ as in Lemma 2.1, or Lemma 2.2 with $q \geq 1$, exists. The residue part of $\tau(E_i)$ is supported on the parallel class of D_i, and its coefficients in that class sum to zero.*

Proof. The free element $\alpha_i E_i$ is closed, since $d(\alpha_i E_i) = -\alpha_i^2 \overline{\alpha}_i = 0$, and its image in A is zero. Agreement of cohomology in degree two gives $[\alpha_i]\tau(E_i) = 0$. Write the residue part of $\tau(E_i)$ as $\rho_i = \sum_j c_{ij} e_j$. Taking type $(2,1)$ gives $[\alpha_i]\rho_i = 0$. No boundary enters this type in degree two by (3). Since the one-residue summands are a direct sum by divisor labels, we obtain

$$c_{ij}\, \alpha_i|_{D_j} = 0 \qquad \text{for every } j.$$

If $\alpha_i|_{D_j} = 0$, the tangent hyperplane of H_j is contained in $\ker \alpha_i$, hence equals it. A connected subgroup of an abelian variety is determined by its Lie algebra, so $H_j = H_i$. Therefore all nonparallel coefficients vanish. The remaining generators have the same nonzero differential by (5). Since ρ_i is closed, their coefficients sum to zero. This argument takes type components of an equality; it never assumes that τ preserves the bigrading. □

4. Circuits obstruct full and partial formality

A circuit of normal lines is a minimally dependent set of distinct lines. Its cardinality is at least three. Choose a circuit of size $r+1$ and representatives so that

$$\alpha_0 = \sum_{i=1}^{r} c_i \alpha_i, \qquad c_i \neq 0,$$

where $\alpha_1, \dots, \alpha_r$ are independent. Put $a_i = c_i \alpha_i$ and $b_i = \overline{c_i}\overline{\alpha}_i$. Multiply every residue symbol in parallel class i by $|c_i|^2$. After this invertible change of variables,

$$dE_i = a_i b_i \ (1 \leq i \leq r), \qquad dE_0 = (a_1 + \cdots + a_r)(b_1 + \cdots + b_r).$$

The corresponding incidence relations still hold, and within each parallel class the normalized coefficient sum still detects residues.

Set $\widehat{a}_i = a_1 \cdots \widehat{a_i} \cdots a_r$ and define in the free Sullivan source

$$c_r = \sum_{i=1}^{r} (-1)^{r-i} \widehat{a}_i E_i - \widehat{a}_r E_0. \tag{7}$$

If $\omega = a_1 \cdots a_r$, the differential of the ith summand is $(-1)^{r-1}\omega b_i$ and that of the last term is $(-1)^r \omega \sum_i b_i$. Hence $dc_r = 0$ in the free algebra.

Theorem 4.1 (Circuit bound)**.** *If the distinct normal lines contain a circuit of size c, then X is not $(c-2)$-formal. In particular, dependent normal lines obstruct formality.*

Proof. Write $c = r+1$ and let $C_r = f(c_r)$. It has degree r and type $(r,1)$. Its coefficient at the chosen divisor of direction 1 is $(-1)^{r-1} a_2 \cdots a_r$, which is nonzero in $\Lambda^{r-1}(H^{1,0}(T)/\mathbb{C}a_1)$. Thus C_r is nonzero by (4) and is not a boundary.

For a parallel class λ, ambient restriction identifies every $H^{r-1,0}(D_j)$ in that class with $\Lambda^{r-1}(H^{1,0}(T)/L_\lambda)$. This identification is canonical on cohomology, since translations act trivially. Define σ_λ to sum coefficients in the class. There are no boundaries in type $(r,1)$, so it detects cohomology classes in that type. In particular, $\sigma_1(C_r) = (-1)^{r-1} a_2 \cdots a_r \neq 0$.

Suppose that X were $(r-1)$-formal. Lemma 2.2 gives one map τ which agrees on cohomology through degree r. It fixes all ambient closed degree-one generators. The type $(r,1)$ part of $\tau(c_r)$ can only use the residue parts of $\tau(E_i)$; the other $r-1$ factors in each summand are actual holomorphic ambient classes. Proposition 3.1 says that every such residue has zero coefficient sum in its parallel

class. Hence every $\sigma_\lambda(\tau(c_r)_{r,1})$ vanishes. This contradicts $H^r(\tau)([c_r]) = [C_r]$. Formality implies $(r-1)$-formality, so it is excluded as well. □

Remark 4.2. If the shortest circuit has length c, the theorem is an upper bound on partial formality. We have not yet determined whether the absence of circuits of length at most $q+2$ is sufficient for q-formality. The proof works even when the selected circuit divisors have empty common intersection, and even when other layers are disconnected: the detected sector has exactly one residue.

5. Independent normals and the isogeny cover

Suppose the normal lines are independent. Then

$$\Phi = (\pi_1, \dots, \pi_s) : T \longrightarrow P = \prod_{\lambda=1}^{s} E_\lambda$$

has surjective differential, and is therefore surjective. Let $K = (\ker \Phi)^0$. By Poincaré complete reducibility [1, Theorem 7.6.1], there is an abelian subvariety $C_0 \subset T$ such that addition $K \times C_0 \to T$ is an isogeny. The restriction $\varphi = \Phi|_{C_0} : C_0 \to P$ is also an isogeny.

Choose a positive integer d annihilating $\ker \varphi$. Multiplication by d on C_0 factors through φ, giving a homomorphism $\psi : P \to C_0$ with $\varphi\psi = [d]_P$. Explicitly, for $z \in P$ choose $c \in C_0$ with $\varphi(c) = z$ and put $\psi(z) = dc$; the choice is immaterial, and local covering charts make the map holomorphic. The map

$$p : K \times P \longrightarrow T, \qquad p(k, z) = k + \psi(z)$$

is an isogeny and satisfies $\Phi p = [d]_P$. Consequently,

$$p^{-1}(X) = K \times \prod_\lambda (E_\lambda \setminus [d]^{-1} S_\lambda).$$

This is a connected finite Galois cover. Its deck group is $\ker p$, acting by translations, and its quotient is exactly X.

An abelian variety is formal (for example by the Kähler formality theorem [2]), and a punctured elliptic curve retracts onto a finite wedge of circles and is formal. Products preserve formality. A finite simplicial group quotient of a formal complex with finite Betti numbers is formal [6, Proposition 2.8]. Triangulate X and lift the triangulation to its finite cover; the deck action is simplicial, so this theorem applies. Scalar-extension invariance returns the conclusion to $\mathbb{Q}$.

The universal cover of K is a complex vector space; punctured elliptic curves are aspherical. Their product is therefore aspherical, and a cover and its base have equal higher homotopy groups. This proves all assertions of Theorem 1.1 except the poset interpretation, which we now spell out.

Corollary 5.1. *The complement is formal exactly when the number of parallel classes equals the maximum layer codimension. For a graphical elliptic arrangement,*

$$\mathrm{Conf}(E, \Gamma) \text{ is formal} \quad \Longleftrightarrow \quad \Gamma \text{ is a forest.}$$

Proof. Two distinct parallel fibers are disjoint. Two nonparallel divisors have independent normal lines, so their map to the product of elliptic quotients is surjective, and all pairs of fibers meet. Thus the poset detects parallel classes. An independent set of ρ normals similarly gives a surjective map to ρ elliptic factors, whose fiber components have codimension ρ. No intersection has larger codimension than the full normal rank. For graph characters $z_i - z_j$, independence of the directions is precisely the absence of graph cycles. □

Corollary 5.2 (Integer-character specialization)**.** *Let E be an elliptic curve and remove finitely many distinct connected divisors $\chi_v^{-1}(p)$ from E^n, where $\chi_v(z) = \sum v_k z_k$ and $v \in \mathbb{Z}^n$ is primitive. The complement is formal exactly when representatives of its distinct rational directions are linearly independent. If all nonempty intersections are connected, a formal complement is transformed by a $\mathrm{GL}_n(\mathbb{Z})$ change of coordinates into a product of punctured elliptic curves and E^{n-s}.*

Proof. The normal lines are spanned by the integer direction vectors, so the first claim is Theorem 1.1. In the second claim, one vector from each independent direction defines a connected common kernel. Smith normal form says that the corresponding rank-s row lattice is primitive; its basis extends to a basis of $\mathbb{Z}^n$, yielding the asserted coordinates. Nonprimitive character equations may first be split into connected fibers, without changing the complement. □

This is the complete range of Pagaria's Question 4.10. The general abelian theorem also permits elliptic quotients which are not mutually isomorphic and complex-multiplication directions which are not represented by one ordinary integer matrix.

6. All group ranks in the formal class

From now on the normal lines are independent. Put $d = n - s$, $r_\lambda = m_\lambda + 1$, and $G = \pi_1(X)$. For a group, write $\gamma_k G$ for its lower central series and set

$$\phi_k(G) = \dim_{\mathbb{Q}}(\gamma_k G/\gamma_{k+1}G) \otimes \mathbb{Q}, \qquad \theta_k(G) = \dim_{\mathbb{Q}} \operatorname{gr}_k(G/G'') \otimes \mathbb{Q}.$$

These are rational ranks; no assertion of integer torsion-freeness is made.

Proposition 6.1. *There are rational subspaces $W, V_1, \dots, V_s$ with $\dim W = 2d$, $\dim V_\lambda = r_\lambda$, such that*

$$H^1(X;\mathbb{Q}) = W \oplus \bigoplus_\lambda V_\lambda, \tag{8}$$

$$\ker(\Lambda^2 H^1(X) \to H^2(X)) = \bigoplus_\lambda \Lambda^2 V_\lambda, \tag{9}$$

$$R_X := \langle H^1(X;\mathbb{Q})\rangle \cong \Lambda W \otimes \bigotimes_\lambda (\mathbb{Q} \oplus V_\lambda), \qquad V_\lambda V_\lambda = 0. \tag{10}$$

In particular, $\operatorname{Hilb}(R_X, t) = (1+t)^{2d} \prod_\lambda (1 + r_\lambda t)$.

Proof. Choose a homomorphism $q : T \to K$ whose restriction to K is an isogeny. To obtain it, take a reverse isogeny of $K \times C_0 \to T$ and compose with projection to K. The homomorphism $F = (q, \Phi) : T \to K \times P$ has injective differential: a vector killed by $d\Phi$ lies in $\operatorname{Lie} K$, where dq is invertible. Equal dimensions make F an isogeny. With

$$Z = K \times \prod_\lambda (E_\lambda \setminus S_\lambda),$$

we have $X = F^{-1}(Z)$, so $f = F|_X : X \to Z$ is a finite covering. Transfer shows that f^* is injective in every rational cohomology degree.

The degree-one Gysin sequence is

$$0 \longrightarrow H^1(T) \longrightarrow H^1(X) \longrightarrow \bigoplus_j \mathbb{Q}[D_j] \longrightarrow H^2(T).$$

The last map has rank s. Indeed, the classes within a parallel family agree, and the s different classes are independent: over $\mathbb{C}$, extend the independent α_λ to a holomorphic basis and use the distinct diagonal wedges $\alpha_\lambda \overline{\alpha}_\lambda$. Hence

$$b_1(X) = 2n + \sum_\lambda m_\lambda - s = 2d + \sum_\lambda r_\lambda = b_1(Z).$$

Thus f^* is an isomorphism in degree one. The cohomology of Z is generated in degree one, and Künneth gives exactly the algebra in (10). The image of f^* is therefore precisely R_X. Its injectivity also identifies the degree-two cup kernel as in (9). □

Let $\mathfrak{h}(G)$ be the cohomological holonomy Lie algebra: the free Lie algebra on $H^1(X;\mathbb{Q})^*$ modulo the image dual to the cup product. By (9),

$$\mathfrak{h}(G) \cong W_{\mathrm{ab}}^* \oplus \bigoplus_{\lambda} \mathrm{Lie}(V_\lambda^*). \tag{11}$$

Here W_{ab}^* is concentrated in degree one and different direct summands commute. Formality implies 1-formality; the standard Malcev comparison identifies $\mathrm{gr}(G)\otimes\mathbb{Q}$ with $\mathfrak{h}(G)$, and identifies the Chen Lie algebra with its metabelian quotient [5, Theorem 4.2 and Corollary 4.3], [11, Proposition 7.6]. These inputs require neither degree-one generation of full cohomology nor unimodularity.

Theorem 6.2. *Let $w_k(r) = k^{-1}\sum_{e|k}\mu(e)r^{k/e}$ be the Witt number. For every formal complement in Theorem 1.1,*

$$\begin{aligned} \phi_1(G) = \theta_1(G) &= 2d + \sum_\lambda r_\lambda, \\ \phi_k(G) &= \sum_\lambda w_k(r_\lambda), && k \geq 2, \\ \theta_k(G) &= \sum_\lambda (k-1)\binom{r_\lambda + k - 2}{k}, && k \geq 2. \end{aligned}$$

Proof. The free Lie algebra on r generators has enveloping algebra the tensor algebra, of Hilbert series $(1-rt)^{-1}$. PBW and logarithmic Möbius inversion give the Witt formula. The abelian summand in (11) contributes only in degree one. For completeness, the free metabelian Lie algebra has degree-k derived part equal to the kernel of the Koszul map

$$V \otimes \mathrm{Sym}^{k-1} V \longrightarrow \mathrm{Sym}^k V, \qquad k \geq 2.$$

Exactness of the polynomial Koszul complex gives its dimension as $r\binom{r+k-2}{k-1} - \binom{r+k-1}{k} = (k-1)\binom{r+k-2}{k}$. This is also the classical free-group Chen formula [5]. Derived and second derived ideals of a direct sum decompose componentwise, so the dimensions add as asserted. □

7. The arithmetic boundary: Koszulness and rational asphericity

We call the arrangement *unimodular* in this section when all its nonempty divisor intersections are connected. This terminology is applied only to the specified geometric arrangement, without assuming an ambient integer-character presentation. Write $P_X(t) = \sum_j b_j(X)t^j$.

Theorem 7.1. *For a formal complement as above, the following conditions are equivalent:*

1. *all nonempty divisor intersections are connected;*
2. $\ker\Phi$ *is connected;*
3. $H^*(X;\mathbb{Q})$ *is generated by* $H^1(X;\mathbb{Q})$*;*
4. $H^*(X;\mathbb{Q})$ *is Koszul in its cohomological grading;*
5. X *is a rational* $K(\pi,1)$ *in the sense of Papadima–Yuzvinsky;*
6. $P_X(-t) = \prod_{k\geq 1}(1-t^k)^{\phi_k(G)}$.

Here the rational $K(\pi,1)$ condition means that the Bousfield–Kan rational completion is aspherical, equivalently that the 1-minimal model is the full minimal model. It is stronger than ordinary asphericity in this setting.

Proof. Selecting one divisor from each parallel direction gives a nonempty fiber of the surjective map Φ, hence a translate of its kernel. This proves (1)⇒(2). Conversely, the kernel of any partial product map is an extension of $\ker\Phi$ by the omitted elliptic factors and is connected when $\ker\Phi$ is connected. Every nonempty intersection is such a fiber; two distinct members of a parallel class have empty intersection. Thus (2)⇒(1).

The Witt identity and Theorem 6.2 give

$$\prod_{k\geq 1}(1-t^k)^{\phi_k(G)} = (1-t)^{2d}\prod_{\lambda}(1-r_\lambda t) = \mathrm{Hilb}(R_X,-t). \tag{12}$$

Since R_X is a graded subspace of $H^*(X)$ with finite dimensions in each degree, equality of its Hilbert series with $P_X(t)$ is equivalent to $R_X = H^*(X)$. Consequently (3) and (6) are equivalent.

We prove that (6) implies (2) without computing the higher-layer model. Set $\overline{T} = T/K$, where $K = (\ker\Phi)^0$. Then Φ factors through an isogeny $\psi : \overline{T} \to P$ of degree $\delta = |\pi_0 \ker\Phi|$. Put $Y = \psi^{-1}(\prod_\lambda(E_\lambda \setminus S_\lambda))$. The real torus sequence $K \to T \to \overline{T}$ splits: the corresponding integer lattice sequence has free quotient, and a lattice section extends to a real linear section. It yields a diffeomorphism $X \cong K \times Y$, so $P_X(t) = (1+t)^{2d}P_Y(t)$. If (6) holds, cancel $(1+t)^{2d}$ in the polynomial equality obtained from (12), to get $P_Y(t) = \prod_\lambda(1+r_\lambda t)$. On the other hand the finite covering $Y \to \prod_\lambda(E_\lambda \setminus S_\lambda)$ gives

$$\chi(Y) = \delta(-1)^s\prod_{\lambda} m_\lambda.$$

Evaluating the polynomial equality at $t=-1$ gives the same expression with δ omitted. Its product is nonzero, so $\delta = 1$ and (2) holds. If $s = 0$, the conclusion is immediate with $\Phi : T \to \{0\}$.

If (2) holds, the same real torus splitting gives an actual diffeomorphism

$$X \cong K \times \prod_{\lambda}(E_\lambda \setminus S_\lambda).$$

Its cohomology is the algebra in (10), proving (3). This algebra is Koszul: an exterior algebra is Koszul; for a square-zero algebra $\mathbb{Q}\oplus V$ the reduced bar complex has zero differential and its length equals internal degree, so its Tor is diagonal. Tensor products of Koszul algebras are Koszul by the tensor-product resolution. Thus (2) implies (4). Conversely a connected Koszul algebra is generated in degree one, since Tor_1 records its indecomposables, so (4) implies (3).

Finally Papadima–Yuzvinsky's theorem for formal spaces of finite rational Betti type states precisely (4)⇔(5) [7, Theorem 5.1]. Its finiteness and formality assumptions hold here. This completes all equivalences. □

Example 7.2 (Ranks do not see disconnected intersections)**.** For $N \geq 2$ let

$$X_N = E^2 \setminus (\{x=0\}\cup\{x+Ny=0\}).$$

The two normals are independent, so X_N is formal and an ordinary $K(\pi,1)$. The map $(x,y)\mapsto(x,x+Ny)$ restricts to an N^2-sheeted cover of $(E\setminus\{0\})^2$. Its two deleted divisors meet in N^2 points. Proposition 6.1 gives

$$\mathfrak{h}(G_N) \cong \mathrm{Lie}(\mathbb{Q}^2)\oplus\mathrm{Lie}(\mathbb{Q}^2), \qquad \phi_1 = 4, \quad \phi_k = 2w_k(2), \quad \theta_k = 2(k-1)\ (k\geq 2).$$

Every X_N has these same rational ranks. Each punctured elliptic curve deformation retracts onto a finite graph. The product retraction lifts through the finite covering, so X_N has the homotopy type of a finite cover of a product of two graphs, a CW complex of dimension at most two. The covering Euler characteristic is N^2 and $b_1 = 4$, yielding

$$P_{X_N}(t) = 1+4t+(N^2+3)t^2, \qquad \mathrm{Hilb}(R_{X_N},t) = 1+4t+4t^2.$$

The excess $(N^2-1)t^2$ consists of cohomology not generated in degree one. The space is not a rational $K(\pi,1)$ despite being formal and ordinarily aspherical. This gives a concrete boundary for any attempted passage from a finite product cover to rational asphericity.

8. Scope of the answer

The normal-circuit obstruction and isogeny construction give a classification for all translated codimension-one connected subgroup arrangements in complex abelian varieties. In the original integer-character elliptic range this answers Question 4.10 of [8]; it answers the formality portion of Problem 8.3 of [9]. The rank and rational-asphericity theorems apply throughout the formal class under the same geometric hypotheses as the classification. The circuit bound supplies only a necessary condition for partial formality. Higher resonance, a complete q-formality classification, and rational $K(\pi, 1)$ for nonformal complements are separate questions. The theorem statements are about the declared geometric class; they do not extend Poincaré complete reducibility to nonprojective complex tori. No claim of exhaustive publication-priority verification is made.

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