# FIBRATIONS, MONODROMY, AND SECTIONS OF ABELIAN ARRANGEMENTS

## BIBBY–DELUCCHI'S ARBITRARY-RANK CONJECTURE IN REMARK 3.3.4 AND MONODROMY QUESTION 5.3.8

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Abstract. We prove that an $M$-ideal of any rank in the layer poset of an abelian arrangement determines a canonical quotient whose complement projection is a smooth fiber bundle. The result includes affine translations and nonessential arrangements. Layer images and bounded horizontal flows prove local triviality even for noncompact fibers, answering the conjecture in Bibby–Delucchi's Remark 3.3.4. For admissible rank-one projections, we then compute integral cohomological monodromy for every connected abelian Lie group, answering their Question 5.3.8. Noncompact fibers have only puncture permutations; compact punctured tori additionally have explicit relative-track shears. In the compact integer-character case the whole representation factors through ambient winding, and its image and kernel are determined by an integer lattice and rational slopes. Finally, we construct sections for arbitrary-rank noncompact split projections and classify sections in a specified compact family with one horizontal isogeny equation. In dimension two the criterion uses the complete finite isogeny kernel and is equivalent to splitting the fundamental-group extension. For scalar isogenies in every compact dimension at least two, a section exists exactly when at most one coefficient is nondivisible by the scalar multiplier. In dimensions at least three this obstruction can persist even when the fundamental-group extension splits.

## 1. Existence, transport, and splitting

Every connected abelian Lie group is isomorphic to $G = \mathbb{T}^d \times \mathbb{R}^v$. We write $m = d+v$ and initially assume $m > 0$. For a free abelian group $\Gamma \cong \mathbb{Z}^n$, the ambient group is $T = \mathrm{Hom}(\Gamma, G) \cong G^n$. In the central setting of [2], a character $\alpha \in \Gamma \setminus \{0\}$ defines $H_\alpha = \{t : t(\alpha) = 0\}$; its connected components are the atoms of the layer poset. We also consider connected affine walls $\{t : t(\lambda) = \xi\}$, with $\lambda$ primitive, and finite collections of such walls. This allows arbitrary translations and selection of connected components. Repeated walls are removed.

Bibby–Delucchi prove the corank-one fibration theorem and conjecture in [2, Remark 3.3.4] that an $M$-ideal of any rank gives a bundle with affine abelian-arrangement fiber. Our first main theorem proves that statement, using the canonical direction determined by the ideal. Their Question 5.3.8 asks for the action of the base fundamental group on the fiber cohomology when an admissible projection is a bundle. Here "admissible" has the specific meaning of [2, Definition 3.1.6]: $\Gamma = \mathbb{Z}\beta_0 \oplus \Gamma''$ and the kernel is the $G$ factor dual to $\mathbb{Z}\beta_0$. It is a rank-one factor, even though $G$ may have arbitrary dimension.

Bibby–Cohen–Delucchi [1] construct the colored and unordered configuration-space pullbacks for corank-one abelian arrangements, describe the toric Artin action, and prove section results in the noncompact case. We use puncture tracks to compute the integral cohomology action. For compact fibers, relative arcs record the shear as well as the puncture permutation. We then study sections in the family with one horizontal isogeny equation. Its finite kernel controls the complete

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two-dimensional criterion; for scalar isogenies a cohomological obstruction gives a classification in every dimension at least two.

For complex hyperplane arrangements, arbitrary-rank modular fibrations were established by Falk–Proudfoot [4]. The proofs below treat affine abelian arrangements through their connected layers and construct horizontal lifts with a uniform bound, allowing noncompact fibers.

**Theorem 1.1** (Arbitrary-rank modular fibrations)**.** *Let $\mathcal{D}$ be a finite affine abelian arrangement in $G^n$, and let $Q$ be an $M$-ideal of rank $q$ in its layer poset $P$. All maximal layers of $Q$ have the same connected direction $Y$. Its annihilator is a primitive rank-$q$ lattice $L \subseteq \Gamma$, and the quotient $p : T \to T/Y \cong \mathrm{Hom}(L, G)$ restricts to a smooth fiber bundle of arrangement complements. Every layer projects to a layer of the base arrangement. Each fiber is the complement of a finite affine abelian arrangement in $G^{n-q}$, and local trivializations preserve all labeled walls.*

**Theorem 1.2** (Integral rank-one transport)**.** *Suppose the admissible rank-one projection in Question 5.3.8 is a bundle, and work on a nonempty connected base component. Its fiber is $F = G \setminus P_0$, where $P_0$ is a finite point set. Its full integral cohomology monodromy is as follows.*

1. *If $v > 0$ and $m \geq 2$, the ambient exterior-algebra classes are fixed and the peripheral classes in degree $m-1$ are permuted by the puncture covering. There is no ambient-peripheral shear. For $G = \mathbb{R}$ the action on $H^0$ is the identity.*
2. *If $G = \mathbb{T}^d$, $d \geq 2$, and $P_0 \neq \varnothing$, the only possibly varying degree is $d-1$. Under the relative-homology identification its action has the form*
$$(a, z) \longmapsto (a + \textstyle\sum_i z_i v_i, \sigma_* z), \quad a \in \mathbb{Z}^d, \quad z \in \mathrm{Aug}\, \mathbb{Z}[P_0],$$
*where $\sigma$ is the puncture permutation and $v_i$ is the closed-cycle class obtained from the $i$th track and chosen reference arcs. For a pure loop the shear is $\sum_i z_i[\tau_i]$.*
3. *For $G = \mathbb{T}$ the action is the cyclic-order-preserving permutation of the complementary intervals, on $H^0$. Empty puncture sets give the identity action.*

*These statements determine every degree and the integral cohomology-ring action. For the original integer-character arrangements, the permutations, shears, and image are explicitly computed from ambient winding in Section 8.*

The section theorem uses $G = \mathbb{T}^2$, $P_* = G \setminus \{0\}$, an integer matrix $C$ with $\det C \neq 0$, arbitrary integer matrices $B_i$, and any $t \in G$. Put

$$\mathcal{M} = \{(b, y) \in G^{q+1} : b_i \neq 0,\ Cy + \textstyle\sum_i B_i b_i \neq t\}, \qquad B = P_*^q. \tag{1.1}$$

Write $D = \mathbb{Z}^2/C\mathbb{Z}^2$, $\beta_i = B_i \bmod C\mathbb{Z}^2$, and $H = \sum_i \mathrm{im}\, \beta_i \leq D$.

**Theorem 1.3** (The complete single-isogeny family)**.** *For all the parameters in (1.1), the projection $\mathcal{M} \to B$ is a smooth bundle. It has a continuous section if and only if it has a smooth section, and these are equivalent to*

$$\#\{i : \beta_i \neq 0\} \leq 1 \quad \text{or} \quad H \text{ is cyclic.} \tag{1.2}$$

*The fundamental-group extension splits exactly under the same condition. Every member acquires a section on an explicit finite base cover. For fixed $q \geq 2$, all matrices $B_i$ and phases admit sections if and only if $D$ is cyclic, equivalently the gcd of the four entries of $C$ is one.*

Arbitrary $2 \times 2$ matrices define real two-torus wall families. When they are lattice matrices of holomorphic endomorphisms of the same elliptic curve, (1.1) is an elliptic arrangement.

## 2. Layers and the canonical direction of an ideal

For a nonempty connected intersection layer $X$, let $U_X \subseteq \Gamma \otimes \mathbb{Q}$ be its rational normal space. Smith normal form gives the identity direction

$$X^0 = \mathrm{Hom}(\Gamma/S^{\mathrm{sat}}, G), \qquad S^{\mathrm{sat}} = (S \otimes \mathbb{Q}) \cap \Gamma,$$

when its equations have row lattice $S$. Thus $\dim X = m(n - \dim_{\mathbb{Q}} U_X)$. We order layers by reverse containment and set $\operatorname{rank} X = \dim_{\mathbb{Q}} U_X$. For a fixed layer $Z$, translate a point of it to the identity. The interval $P_{\leq Z}$ is the geometric lattice of the incident rational normals: the component through that point is uniquely determined by their saturated span. In particular its rank is the rank just specified.

**Definition 2.1** ($M$-ideal)**.** A nonempty subset $Q \subseteq P$ is an $M$-ideal if it is a pure order ideal, is join-closed, and has the following properties. Join-closed means that every minimal common upper bound of any subset of $Q$, when such bounds exist, belongs to $Q$.

(1) If $y \in Q$ and an atom $a$ have no common upper bound, then $a \in Q$.
(2) For each maximal $z \in P$ there is a maximal $y \in Q$ with $y \leq z$ that is modular in $P_{\leq z}$.

All maxima of $Q$ have rank $q$. In a geometric lattice, modularity is equivalently the identity

$$\operatorname{rank} x + \operatorname{rank} y = \operatorname{rank}(x \vee y) + \operatorname{rank}(x \wedge y) \quad \text{for every } x.$$

This is [2, Definition 2.4.1], applied to the affine affine layer poset. All components and phases participate in its second condition.

**Lemma 2.2.** *Fix $y_0 = t_0 + Y \in \max Q$ and let $L = U_{y_0} \cap \Gamma$. The atoms of $Q$ are exactly all walls whose normals lie in $L$. Consequently $Q$ is the layer poset of this vertical subarrangement, and every maximal layer of $Q$ has direction $Y$.*

*Proof.* $L$ is primitive of rank $q$, so choose $\Gamma = L \oplus K$. Then $Y = \operatorname{Hom}(K, G)$, and $T = \operatorname{Hom}(L, G) \times \operatorname{Hom}(K, G)$. A nonzero integer row on $G^k$ is onto $G$: integer unimodular operations reduce it to multiplication by a nonzero integer, and that multiplication is onto $\mathbb{T}^d \times \mathbb{R}^v$. If the primitive normal $\lambda$ of a wall $a$ is not in $L$, its restriction to $Y$ is such a nonzero row. The wall therefore meets $y_0$ but does not contain it. If $a \in Q$, a component of that intersection is a common upper bound strictly above $y_0$, and join-closedness contradicts maximality of $y_0$. If $\lambda \in L$, its value on $y_0$ is constant. The wall either contains $y_0$, when the order-ideal property puts it in $Q$, or is disjoint from $y_0$, when condition (1) of Definition 2.1 puts it in $Q$.

Every layer is generated by atoms in its local geometric interval. Conversely every connected component of an intersection of vertical atoms is a minimal common upper bound, hence belongs to $Q$. Thus their layer poset is exactly $Q$. All its normal spaces lie in $L \otimes \mathbb{Q}$; purity makes those of its maxima have rank $q$, so they equal that space and have the same direction $Y$. □

Vertical walls are exactly full inverse images of walls under $p : T \to \operatorname{Hom}(L, G)$. Since $Y$ is connected, base connected layers have connected inverse images. This identifies the layer posets $Q \cong P(\mathcal{D}_B)$.

**Lemma 2.3** (Images of layers)**.** *For every layer $X \in P$, $p(X)$ is a base layer. It is either the whole base torus-group or is disjoint from the base complement $B = M(\mathcal{D}_B)$.*

*Proof.* Choose a maximal $z \geq X$ and a modular $y \in \max Q$ in its local interval. Set $w = X \wedge y$. Then $w \in Q$, and the join through $z$ has normal space $U_{X \vee y} = U_X + (L \otimes \mathbb{Q})$. Since $w$ contains $X$ and $y$, $U_w \subseteq U_X \cap (L \otimes \mathbb{Q})$. Modularity gives

$$\dim U_w = \operatorname{rank} X + q - \operatorname{rank}(X \vee y) = \dim\big(U_X \cap (L \otimes \mathbb{Q})\big),$$

so equality holds between those normal spaces.

$w$ is vertical, with $w = p^{-1}(p(w))$, and $X \subseteq w$. The tangent image of $X$ under $dp$ has annihilator $(U_X \cap (L \otimes \mathbb{Q})) \otimes (\mathbb{R}^m)^*$ in the base cotangent space. It is therefore exactly the tangent space of $p(w)$. Translate a common point of $z$ to the identity. The restriction from the connected direction of $X$ to the connected direction of $p(w)$ is a Lie homomorphism with surjective derivative. Its image is an open subgroup; a connected group has no proper open subgroup. Hence $p(X) = p(w)$ as cosets. If $w = T$ this is the whole base; otherwise $p(w)$ lies in a base wall and misses $B$. □

If the total normal rank is $r < n$, its saturated lattice $\Lambda$ splits off in $\Gamma$. The arrangement pair is exactly its essential pair in $\mathrm{Hom}(\Lambda, G)$ times $G^{n-r}$, with every phase retained. Every maximal layer has rank $r$: if its rank were smaller, a wall normal outside its normal space would restrict nontrivially, hence surjectively, to its direction and would meet that layer, increasing its rank. The canonical kernel includes the passive factor.

## 3. Bounded horizontal lifting without properness

**Lemma 3.1** (Simultaneous local trivialization)**.** *Let finitely many closed affine integer walls lie in $G^n$, and let $p : G^n \to G^q$ be a split projection. Suppose that, over an open set $B$, every intersection direction present there maps onto the base tangent space. Then around every point of $B$ the ambient projection admits a smooth trivialization preserving every labeled wall.*

*Proof.* Use a complete flat metric on $G^n$ and identify its tangent spaces with $E$. Write $P : E \to E_B$ for $dp$. Only finitely many intersection directions occur. For each such direction $V$, choose a linear right inverse $j_V : E_B \to V$, and include the full ambient direction. Then $C_0 = \max_V \|j_V\| < \infty$. Choose a convex group-coordinate neighborhood $W \subseteq B$ of $b_0$. For each $x \in p^{-1}(W)$ choose $O_x$ avoiding all walls not containing $x$, with $O_x \subseteq p^{-1}(W)$. This is possible since the wall list is finite and closed. Let $V_x$ be the direction of the intersection of all incident walls, or $E$ if none are incident. By hypothesis it maps onto $E_B$. Take a smooth locally finite partition of unity $\rho_a$ subordinate to these neighborhoods and set

$$J(z)u = \sum_a \rho_a(z) j_{V_{x_a}} u.$$

Then $PJ = \mathrm{id}$ and $\|J(z)\| \le C_0$. If $z$ is on a wall and $\rho_a(z) \neq 0$, that wall must contain $x_a$, so the corresponding lift lies tangent to it. Therefore $J(z)u$ is tangent to every incident wall.

For $b \in W$, let $u_b = b - b_0$ in these coordinates and consider the smooth field $H_b(z) = J(z)u_b$. Starting over $b_0$, its base trajectory is $b_0 + t(b - b_0)$ and its speed is at most $C_0\|b - b_0\|$. For $0 \le t \le 1$ the trajectory stays in a closed bounded ambient ball and over this compact line segment. Their intersection is a compact subset of the field domain $p^{-1}(W)$: the ambient complete flat metric is proper. A solution with a finite maximal endpoint in $[0, 1]$ would remain in this compact subset, where the field is smooth, and hence would extend across that endpoint by the ODE continuation theorem. The time-one flow therefore exists for every initial point. The time-minus-one flow from the fiber over $b$ has the same bounds and exists as well.

Uniqueness makes these flows inverse. Smooth dependence on initial points and $b$ gives a smooth map $p^{-1}(b_0) \times W \to p^{-1}(W)$ and its smooth inverse. Each wall is an embedded closed submanifold tangent to the field. Its restricted ODE has the same solutions; both time directions therefore preserve it. This proves simultaneous trivialization of the entire labeled pair. □

*Proof of Theorem 1.1, local statement.* Lemmas 2.2 and 2.3 give the asserted kernel and base arrangement. At a point over $B$, the intersection component of all incident walls is a layer meeting $p^{-1}(B)$, so its projection is the whole base and its tangent projection is onto. Lemma 3.1 therefore applies. Deleting the walls gives the bundle trivialization. In the integer splitting $\Gamma = L \oplus K$ the fiber equations are exactly

$$y(\lambda_K) = \xi - b(\lambda_L).$$

Vertical walls do not occur over $B$, and every remaining $\lambda_K$ is nonzero. Its finite component equations give an affine abelian arrangement in $G^{n-q}$. The fiber is nonempty: finitely many positive-codimension closed submanifolds cannot fill a positive-dimensional ambient group. If $n = q$, there are no horizontal walls and the fiber is a point. □

3.1. **One fiber type across disconnected real bases.** When $m \geq 2$, $B$ is path connected: a path in the connected base group can be perturbed relative to its endpoints to miss finitely many walls of codimension $m \geq 2$. Local transport gives one fiber type. For $m = 1$, however, $G = \mathbb{R}$ or $\mathbb{T}$, and base components must also be considered.

**Lemma 3.2.** *In Theorem 1.1 all base components have diffeomorphic fibers. If $G = \mathbb{R}$ they are a fixed number of copies of $\mathbb{R}^{n-q}$. If $G = \mathbb{T}$ they are a fixed number of copies of $\mathbb{R}^{r-q} \times \mathbb{T}^{n-r}$, where $r$ is the total normal rank.*

*Proof.* For any fixed subset of horizontal walls, its intersection is a finite union of layers. Each layer image is either the whole base or disjoint from all of $B$, by Lemma 2.3. In the first case each fiber intersection is a translate of the same kernel of a surjective homomorphism of connected directions. Its dimension and component number are thus constant throughout $B$, including across distinct components.

For $G = \mathbb{R}$, the fiber is a finite affine hyperplane complement. Each chamber is an open convex polyhedron in $\mathbb{R}^k$, $k = n - q$, and is diffeomorphic to $\mathbb{R}^k$. One explicit justification is to write the chamber as $\{\ell_j > 0\}$ and use

$$\phi(x) = \tfrac{1}{2}\|x\|^2 + \sum_j \ell_j(x)^{-1}.$$

Its Hessian is positive definite. For every $u$, $\phi(x) - u \cdot x$ tends to infinity at finite boundary points and at infinity and has a unique minimum. Thus $\nabla\phi$ is a bijection onto $\mathbb{R}^k$ with invertible derivative, giving the stated diffeomorphism. For no constraints the assertion is immediate. Compactly supported Euler characteristic and inclusion–exclusion show that the number of chambers is constant: all subset intersections have the same dimension and component number on all of $B$, and each chamber contributes $(-1)^k$.

For $G = \mathbb{T}$, saturate the span of the horizontal restricted normals in $K$ and split it off. Its rank is $h = r - q$, so the fiber pair is the exact product of an essential arrangement pair in $\mathbb{T}^h$ and the passive torus $\mathbb{T}^{k-h}$. A lifted chamber of the essential arrangement lies between two consecutive integer levels for each normal. Independent normals bound it in $h$ directions, so it is a bounded open convex polyhedron. A nonzero integer translation cannot stabilize it; translates that meet are the same chamber. Hence its projection is injective and onto an entire torus-complement component, and each such component is $\mathbb{R}^h$. There are finitely many, as can be checked in a unit cube. Removing the passive factor does not change the component numbers of the subset intersections. Their Euler characteristics are zero in positive dimension and equal the point counts in dimension zero. The same inclusion–exclusion calculation thus shows the essential chamber number is constant throughout $B$. If $h = 0$ there are no horizontal walls and the fiber is the passive torus. This proves both assertions. □

Together with the local statement this completes the full fiber-bundle assertion of Theorem 1.1.

3.2. **Why the quotient must be canonical.** In $\mathbb{C}^3$ take the three coordinate walls and the rank-one ideal generated by $x_3 = 0$. Its canonical kernel is the plane $\{x_3 = 0\}$. The smaller subgroup $Y' = \{(t, t, 0)\}$ has the same list of walls containing it, but quotient coordinates $(x_1 - x_2, x_3) = (b, c)$ give fibers with two deleted points for $b \neq 0$ and one for $b = 0$, over $c \neq 0$. This is not a bundle. The direction and its dimension in the theorem are therefore essential. Likewise, rational modularity alone cannot replace the conditions on every maximal layer: the toric characters $(1, 0), (0, 1), (1, 2)$ have a puncture collision over the additional phase $z_2 = -1$ when projecting along $z_1$. Their $M$-ideal condition fails there, as discussed in [2, Example 3.1.9].

## 4. The geometric extension to compact complex tori

The canonical quotient also exists for compact complex tori. A supersolvable flag means a chain $\{A\} = Q_0 \subset Q_1 \subset \cdots \subset Q_r = P$ with $\operatorname{rank} Q_i = i$ and $Q_i$ an $M$-ideal in $Q_{i+1}$.

**Theorem 4.1.** *Let $A$ be a compact complex torus of dimension $n$, and let the walls be finitely many distinct translates of connected complex-codimension-one subtori. For an $M$-ideal $Q$ of complex rank $q$, its maximal layers have a canonical connected direction $Y$. The quotient $A \to A/Y$ restricts to a smooth arrangement bundle and every layer projects to a base layer. If the layer poset has a supersolvable flag of $M$-ideals, the complement is a $K(\pi, 1)$.*

*Proof.* Intersections have finitely many components, and the interval at a layer is the geometric lattice of the incident complex hyperplanes. Its rank is complex codimension. Fix a maximal $y_0 = t_0 + Y$ of $Q$. For a wall with direction $D$, the quotient $A/D$ is an elliptic curve. If $Y \nsubseteq D$, the restriction $Y \to A/D$ is a nonzero homomorphism of compact connected complex tori. Its image is a positive-dimensional connected subtorus of the elliptic curve and is therefore the whole curve. Consequently the wall meets $y_0$ properly. Such a wall cannot belong to $Q$: join-closedness would put a component of this intersection strictly above its maximal element $y_0$. If $Y \subseteq D$, the wall either contains $y_0$ or is disjoint from it. The order-ideal property in the first case and condition (1) of Definition 2.1 in the second put that wall in $Q$.

Thus the atoms of $Q$ are precisely the walls whose directions contain $Y$. Their intersections generate $Q$. All maximal elements have codimension $q$ by purity, so their directions, which contain $Y$ and have its dimension, equal $Y$. These walls descend under $p : A \to A/Y$ to connected codimension-one subtori and their translates. The inverse image of each connected base layer is connected, since the quotient has connected kernel $Y$. This identifies $Q$ with the layer poset of the base arrangement. The same surjectivity also shows that all maximal layers have the total complex normal rank $r$.

For a layer $X$, choose a common maximal layer $z$ and a modular $y \in \max Q$ through it. At a point $z_0 \in z$ write $X = z_0 + X_0$ and $y = z_0 + Y$. The meet $w = X \wedge y$ contains $X$ and $y$, so its direction contains $X_0 + Y$. This sum is closed and connected, being the image of the compact connected torus $X_0 \times Y$ under addition. The join in the local interval is the component of $X \cap y$ through $z_0$. Modularity gives

$$\dim_{\mathbb{C}} w = \dim_{\mathbb{C}} X_0 + \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}}(X_0 \cap Y) = \dim_{\mathbb{C}}(X_0 + Y).$$

The inclusion of connected directions of equal dimension is equality. Hence

$$w = z_0 + (X_0 + Y), \qquad p(X) = p(w).$$

This proves the layer-image assertion. Finitely many tangent directions admit right inverses as in Lemma 3.1; its partition-of-unity construction works in the flat tangent bundles of the quotient map. Compactness and the compact base segment ensure both flows exist. The base complement is connected by the codimension-two path argument.

For the last assertion induct on $r$ along the flag. If $r = 0$, the torus itself is aspherical. In a corank-one step let $Z$ be the identity direction common to all walls, of complex dimension $n - r$. The canonical $Y$ has dimension $n - r + 1$ and contains $Z$. Each horizontal wall restricts on $Y$ to finitely many $Z$ cosets, because the identity intersection has dimension $n - r$. The fiber is therefore the inverse image of an elliptic curve minus finitely many points under $Y \to Y/Z$, a bundle with torus fiber $Z$. Both the base of this fiber bundle and $Z$ are aspherical, so its homotopy exact sequence makes the arrangement fiber aspherical. The outer base is aspherical by induction, and its homotopy exact sequence then makes the total complement aspherical. □

This theorem applies to complex-multiplication elliptic equations after decomposing each deleted fiber into its connected components. The quotient construction is intrinsic to the compact complex torus, so it applies to the endomorphism modules studied in [8] without choosing a module basis. For a connected fiber and an aspherical base, the homotopy exact sequence also gives the associated fundamental-group extension.

## 5. Rank-one fibers and the puncture covering

For the admissible rank-one projections of Question 5.3.8, let $\Gamma = \mathbb{Z}\beta_0 \oplus \Gamma''$, $s = \operatorname{rank}\Gamma''$. The ambient projection is the product $G^s \times G \to G^s$. For each horizontal row, change its sign if necessary and write

$$\alpha = c_\alpha \beta_0 + a_\alpha, \qquad c_\alpha > 0, \quad a_\alpha \in \Gamma''. \tag{5.1}$$

For an affine wall write its phase as $\xi_\alpha \in G$; the central case has $\xi_\alpha = 0$. Over the complement of the vertical walls its punctures satisfy

$$c_\alpha y = \xi_\alpha - b(a_\alpha). \tag{5.2}$$

Multiplication by $c$ on $G$ is a covering of degree $c^d$, with kernel $(\mathbb{Z}/c)^d$; the real factor is an invertible linear map. Each row therefore supplies finitely many local smooth root branches.

**Lemma 5.1.** *If the complement projection is a bundle, its set of distinct punctures is a finite covering of each connected base component. The original bundle is exactly the pullback of the unordered universal punctured-$G$ bundle. Under the irredundancy assumption in* [2]*, roots of different rows are disjoint throughout the base.*

*Proof.* Let $\ell(b)$ be the number of distinct punctures. If $v > 0$ and $m \geq 2$, excision and the relative homology sequence give

$$\operatorname{rank} H_{m-1}(G \setminus P_0; \mathbb{Z}) = \binom{d}{m-1} + |P_0|.$$

Indeed $H_m(G) = 0$ and the relative point groups are $\mathbb{Z}[P_0]$ in degree $m$ and zero otherwise. If $v = 0$, $d \geq 2$, the same sequence has $H_d(G) = \mathbb{Z}$ mapping diagonally into $\mathbb{Z}[P_0]$; for nonempty $P_0$ the rank is $d + |P_0| - 1$. An empty puncture set is distinguished by $H_d(G)$. For $G = \mathbb{R}$ or $\mathbb{T}$, the numbers of complementary intervals are respectively $|P_0| + 1$ and $|P_0|$ when punctures are present. An empty circle and a circle with one puncture both have one component, but $H^1$ distinguishes them. These homological invariants determine $\ell(b)$, which is therefore locally constant for a bundle.

Near a point $b_0$, take disjoint balls around its punctures. Shrink the base so each original local root stays in its assigned ball. There is at least one point in every ball; constant total count forces exactly one in each. All roots coinciding at $b_0$ must therefore continue to coincide, and the point set consists locally of disjoint smooth graphs. This is a finite covering and defines a continuous map to $U\operatorname{Conf}_\ell(G) = \operatorname{Conf}_\ell(G)/\mathfrak{S}_\ell$. The pullback of

$$\{(S, y) : S \in U\operatorname{Conf}_\ell(G),\ y \notin S\} \longrightarrow U\operatorname{Conf}_\ell(G)$$

is precisely the original set of pairs $(b, y)$. If two distinct row roots met in the irredundant setting, the local graph argument would make open subsets of their horizontal wall components equal. Equal affine-coset germs have equal directions and phase, so those connected components would coincide globally, contrary to irredundancy. □

The colored pullback is established in [1, Theorem 2.5.1]; its unordered formulation and the relation to the $TM$ condition are given in their Remarks 2.3.3 and 2.5.3. The preceding proof supplies the puncture-cover description used in the transport calculation.

A configuration loop has labeled lifted tracks $\tau_i : p_i \to p_{\sigma(i)}$. It can be represented by smooth tracks. They admit a compactly supported ambient isotopy $h_t$ carrying each initial point to its track: subdivide time into intervals with disjoint small coordinate balls, choose bump fields realizing the point velocities, and integrate. In a coordinate ball around $\tau_i(t)$, multiply its velocity by a bump function equal to one at that point. Disjoint balls make the sum of these fields attain the prescribed velocity at every $\tau_i(t)$. The uniqueness theorem for the ODE then gives $h_t(p_i) = \tau_i(t)$. The union of the tracks and of finitely many chosen supports is compact, so the flows exist for all the required times, also on noncompact $G$. The isotopy starts at the identity and preserves orientation. If $h'_t$

realizes the same tracks, $(h'_t)^{-1}h_t$ fixes the initial point set throughout and restricts to an isotopy of its complement. Hence the induced cohomology map is independent of that choice. We fix forward transport as

$$T_\gamma = (h_1^{-1})^*. \tag{5.3}$$

Using $h_1^*$ instead reverses all the following matrices and permutations.

## 6. Noncompact fibers: the positive end removes all shear

Assume $v > 0$, $m \geq 2$, and let $P_0$ be the puncture set. Excision in disjoint oriented point balls gives

$$H_i(G, G \setminus P_0; \mathbb{Z}) = \begin{cases} \mathbb{Z}[P_0], & i = m, \\ 0, & i \neq m. \end{cases}$$

Since $G \simeq \mathbb{T}^d$ and $m > d$, the homology exact sequence yields

$$0 \longrightarrow \mathbb{Z}[P_0] \longrightarrow H_{m-1}(F; \mathbb{Z}) \longrightarrow \bigwedge^{m-1} \mathbb{Z}^d \longrightarrow 0, \tag{6.1}$$

and $H_i(F) \to H_i(G)$ is an isomorphism in every other degree. Each puncture generator maps to its positive small-sphere class $\delta_p$. All groups are free. If $v \geq 2$, the right group in (6.1) is zero, so no extension can carry a shear. If $v = 1$, that group is $\mathbb{Z}$, and an invariant splitting is needed to determine the action.

**Lemma 6.1** (The end-torus splitting)**.** *For $G = \mathbb{T}^d \times \mathbb{R}$, $d \geq 1$, the class $u_+ = [\mathbb{T}^d \times \{R\}]$ for $R$ above every puncture is independent of sufficiently large $R$. It supplies a splitting*

$$H_d(F; \mathbb{Z}) = \mathbb{Z}u_+ \oplus \bigoplus_{p \in P_0} \mathbb{Z}\delta_p$$

*that is natural under all puncture motions.*

*Proof.* Two such slices bound the cylinder between them, which misses all punctures. They define the same class, and their image in $H_d(G)$ is its standard generator. The class therefore splits (6.1). Along any configuration loop the tracks lie in a compact set. The ambient isotopy described above can be supported in a compact neighborhood of those tracks, so a slice at a sufficiently large $R$ is fixed pointwise. Its class is invariant. The isotopy preserves orientation and sends $\delta_p$ to $\delta_{\sigma(p)}$. These assertions are independent of the chosen extension isotopy, as already proved. □

The ambient action in every other degree is the identity, since $h_t$ is an isotopy on $G$. Universal coefficients, with no Ext term because all homology is free, now give

$$H^j(F; \mathbb{Z}) = \bigwedge^j \mathbb{Z}^d \ \oplus \begin{cases} \mathbb{Z}[P_0]^*, & j = m-1, \\ 0, & j \neq m-1. \end{cases} \tag{6.2}$$

The ambient summand consists of pullbacks from $G$. Let $\varepsilon_p$ be dual to $\delta_p$, and when $v = 1$ require $\varepsilon_p(u_+) = 0$. Lemma 6.1 makes that normalization natural. Under the forward convention (5.3),

$$T_\gamma \varepsilon_p = \varepsilon_{\sigma(p)}. \tag{6.3}$$

All ambient exterior-algebra classes are fixed. $H^j(F) = 0$ for $j > m-1$, so the peripheral summand has square zero and has zero product with every positive-degree class. This proves the entire cohomology-ring statement, including the potentially ambiguous cylinder case $d = v = 1$.

If $m = 1$ and $v > 0$, then $G = \mathbb{R}$. Distinct points cannot change their linear order along a motion. The $\ell + 1$ complementary intervals are preserved, so the action on $H^0(F; \mathbb{Z}) = \mathbb{Z}^{\ell+1}$ is the identity and all other cohomology is zero. The degree-zero ring is the product ring $\mathbb{Z}^{\ell+1}$.

## 7. Compact fibers: relative arcs determine the integral shear

Let $G = \mathbb{T}^d$, $d \geq 2$, and $P_0 = \{p_0, \dots, p_{\ell-1}\} \neq \varnothing$. Deleting small open balls gives a compact manifold $W$ homotopy equivalent to $F$. Poincaré–Lefschetz duality [5, Section 3.3], excision, and contraction of the balls to their centers give a natural identification

$$H^j(F;\mathbb{Z}) \cong H_{d-j}(G, P_0;\mathbb{Z}). \tag{7.1}$$

For an orientation-preserving puncture homeomorphism the duality convention is $\mathrm{PD}((h^{-1})^* x) = h_* \mathrm{PD}(x)$. This specifies the signs and the transport direction. For $j \leq d-2$, inclusion gives the ambient cohomology $H^j(G)$. The only exceptional degree is $d-1$, where the relative sequence is

$$0 \longrightarrow H_1(G;\mathbb{Z}) \longrightarrow H_1(G, P_0;\mathbb{Z}) \longrightarrow \mathrm{Aug}\,\mathbb{Z}[P_0] \longrightarrow 0. \tag{7.2}$$

The augmentation group consists of integer point coefficients whose sum is zero. Both ends are free, but its splittings need not be invariant. The ambient top class has vanished: $H^d(F) = 0$.

Choose reference arcs $\kappa_i : p_0 \to p_i$ with $\kappa_0$ constant, and set $e_i = [\kappa_i]$ in relative homology, so $e_0 = 0$. For tracks $\tau_i : p_i \to p_{\sigma(i)}$, put

$$v_i = [\kappa_i + \tau_i - \kappa_{\sigma(i)}] \in H_1(G;\mathbb{Z}) = \mathbb{Z}^d. \tag{7.3}$$

The bracket contains a closed cycle. Identify (7.2) by the splitting $(a, z) \mapsto a + \sum_i z_i e_i$.

**Proposition 7.1** (The relative-track formula)**.** *In degree $d-1$, the complete forward transport is*

$$T_\gamma(a, z) = \left( a + \sum_i z_i v_i,\ \sigma_* z \right). \tag{7.4}$$

*For a pure loop it is*

$$T_\gamma(a, z) = \left( a + \sum_i z_i [\tau_i],\ z \right). \tag{7.5}$$

*All other degrees are fixed. Changing reference arcs conjugates these matrices by an integral upper-triangular change of splitting.*

*Proof.* The isotopy prism on $\kappa_i$ has boundary

$$h_1(\kappa_i) - \kappa_i - \tau_i + \tau_0.$$

In relative homology this gives

$$h_{1*} e_i = v_i - v_0 + e_{\sigma(i)} - e_{\sigma(0)}.$$

Summing with coefficients $z_i$ of sum zero removes both common terms, and ambient cycles are fixed by the isotopy. Formula (7.1) gives (7.4) with the indicated plus sign. For $\sigma = \mathrm{id}$ the reference arcs cancel from the closed cycle and $v_i = [\tau_i]$. Replacing a reference arc changes $e_i$ by an ambient integral cycle. The resulting splitting change has the asserted form and conjugates the same intrinsic action. Low degrees are fixed because their ambient pullbacks are fixed. □

The cohomology ring is the ambient exterior algebra with its degree-$d$ part removed, together with $\ell - 1$ additional free classes in degree $d-1$, after a choice of splitting. Each additional class has zero product with every positive-degree class, by degree. Thus the displayed formula determines the full ring action. For $d = 2$ all products of positive-degree classes are zero, so the same statement includes every degree-one shear.

**Corollary 7.2** (Finite and trivial actions)**.** *For a compact punctured torus with $d \geq 2$ and $\ell \geq 2$, its entire cohomology action is trivial if and only if every puncture permutation is the identity and every pure loop has equal track homology classes at all punctures. Its image is finite if and only if all pure relative track differences vanish. On the pure subgroup, $(T_\gamma - I)^2 = 0$ and the image is a finitely generated free abelian group.*

*Proof.* Permutation on $\operatorname{Aug}\mathbb{Z}[P_0]$ is faithful: fixing all point differences fixes each point, including the two-point case, where the swap acts by $-1$. The shear in (7.5) is zero exactly when every $[\tau_i]-[\tau_0]$ is zero. Every pure matrix has one off-diagonal block and is the identity on both diagonal blocks. Its nilpotent part $N$ has $N^2=0$, pure matrices multiply by addition of these blocks, and a nonzero $N$ gives $(I+N)^k=I+kN\neq I$ for every nonzero integer $k$. Hence a nonzero pure shear has infinite order. The pure subgroup has finite index, so vanishing on it is also sufficient for finite image. Its image is a subgroup of the finite-rank free abelian group $\operatorname{Hom}(\operatorname{Aug}\mathbb{Z}[P_0],\mathbb{Z}^d)$ and is therefore free abelian and finitely generated. □

For $\ell=1$ the augmentation group is zero and the action is always trivial. For $G=\mathbb{T}$ and $\ell>0$, the complement consists of $\ell$ intervals. Label each by its positive left endpoint. An isotopy preserves cyclic order, so it acts on $H^0(F)=\mathbb{Z}^\ell$ by the same cyclic permutation of punctures; no positive-degree group remains. For no punctures the action on a torus is the identity. If $G$ has dimension zero, its possible fibers are a point or empty and the action is respectively the identity or the zero-module action. These observations and Section 6 complete the proof of Theorem 1.2.

The relative-arc and point-pushing mechanisms have classical antecedents, for example [3, 7]. Here the prism argument provides the closed-torus formula, including nonpure permutations, with the specified integral convention.

## 8. Winding formulas for the original integer-character bundles

Retain (5.1) and the essential, irredundant central hypotheses of [2], so that $\xi_\alpha=0$, for the arithmetic image statements. Lemma 5.1 shows that every row supplies its own disjoint root block. For a loop $\gamma$, denote its ambient torus winding by

$$W\in\operatorname{Hom}(\Gamma'',\mathbb{Z}^d)\cong\mathbb{Z}^{ds}.$$

Choose a root at the base point, and label its translates by $k\in(\mathbb{Z}/c_\alpha)^d$. Lifting (5.2) to the real universal cover gives the exact endpoint permutation

$$\sigma_W(\alpha,k)=(\alpha,k-W(a_\alpha)\bmod c_\alpha). \tag{8.1}$$

Real coordinates of $G$ have unique ordinary lifts and no endpoint winding, so they introduce no additional permutation. Under persistent redundancy one can quotient these labels by the equivalence relation identifying coincident roots; the geometric transport formulas remain valid, although the simple arithmetic classification here uses the original irredundant setting.

8.1. **Noncompact image and the degree-one criterion.** For $v>0$ the cohomology action is exactly (8.1) on peripheral classes, together with ambient identity. Every ambient winding is realized by a base loop. To see this, choose the first real coordinate tuple $r\in\mathbb{R}^s$ so that $a(r)\neq0$ for every vertical row $a$. The complement of finitely many proper real hyperplanes supplies such an $r$. The whole torus $\mathbb{T}^{ds}$ at these fixed real coordinates lies in the base complement and maps isomorphically on ambient fundamental groups. A path from the chosen base point to this torus changes no winding class. Thus the image is the translation-permutation image of the finite homomorphism

$$\mathbb{Z}^{ds}\longrightarrow\bigoplus_\alpha(\mathbb{Z}/c_\alpha)^d,\qquad W\longmapsto(-W(a_\alpha)\bmod c_\alpha)_\alpha. \tag{8.2}$$

Smith normal form computes this image, its kernel, and the point orbits from the integer rows. For $d>0$ the action is trivial exactly when $c_\alpha$ divides every coefficient of $a_\alpha$ for each horizontal row. For $d=0$ every root is globally unique and the action is trivial; this includes the real-line case.

If $g_\alpha$ is the gcd of the coefficients of $\alpha$, a connected horizontal component projects with degree $(c_\alpha/g_\alpha)^d$. Indeed divide the row by $g_\alpha$ and apply the covering degree of multiplication by its remaining kernel coefficient. For $d>0$ the trivial-action criterion is therefore exactly that all these component degrees are one, the geometric $TM$ condition of [2]. This recovers the noncompact statement there while specifying its entire nontrivial permutation representation.

8.2. **Compact winding determines even nonpure shears.** Assume now $G = \mathbb{T}^d$, $d \geq 2$. For each puncture $i$ in row block $\alpha$, choose a lift $x_i \in \mathbb{R}^d$ and use the projected straight segment from $x_0$ to $x_i$ for $\kappa_i$. A lifted root path ends at $x_i - W(a_\alpha)/c_\alpha$, so

$$v_i(W) = x_i - x_{\sigma_W(i)} - W(a_\alpha)/c_\alpha \in \mathbb{Z}^d. \tag{8.3}$$

It is an integer vector because the first and last points project to the same endpoint puncture. The reference arcs close the lifted root displacement into an integer homology class. Equations (7.4), (8.1), and (8.3) are the complete integer matrices of the original representation. They depend only on $W$, and satisfy

$$v_i(W) + v_{\sigma_W(i)}(V) = v_i(W + V),$$

since each permutation stays within a row block. Consequently the whole representation factors through $\mathbb{Z}^{ds}$ and its image is abelian.

The winding map from the base fundamental group to $\mathbb{Z}^{ds}$ is onto: its deleted vertical components have real codimension $d \geq 2$, and an arbitrary ambient loop can be perturbed relative to its base point to miss them. The same argument connects the base. The subgroup killing all root permutations has finite index, so its winding image is a finite-index sublattice of $\mathbb{Z}^{ds}$.

Put $s_\alpha = a_\alpha/c_\alpha \in \Gamma'' \otimes \mathbb{Q}$. For a pure loop all tracks in that block have the same winding $-W(s_\alpha)$. Fix a reference block $\alpha_0$ when horizontal rows exist, and set

$$k = \dim_{\mathbb{Q}} \operatorname{span}\{s_\alpha - s_{\alpha_0}\}.$$

With no horizontal rows, set $k = 0$ and $\mathcal{L} = \mathbb{Z}^{ds}$; the fiber is the unpunctured torus and its action is trivial.

**Theorem 8.1** (Complete compact arithmetic image)**.** *In the original integer-character setting with $G = \mathbb{T}^d$, $d \geq 2$, and nonempty base, the pure cohomology image has free rank $dk$. The entire image is canonically a quotient of the ambient winding lattice by*

$$\begin{aligned} \mathcal{L} = \{W \in \mathbb{Z}^{ds} : {} & W(a_\alpha) \in c_\alpha \mathbb{Z}^d \text{ for every } \alpha, \\ & W(s_\alpha - s_{\alpha_0}) = 0 \text{ for every } \alpha\}, \end{aligned} \tag{8.4}$$

*namely $\mathbb{Z}^{ds}/\mathcal{L}$. It is finite if and only if all horizontal slopes are equal. It is trivial if and only if those slopes are equal and every $c_\alpha$ divides all coefficients of $a_\alpha$.*

*Proof.* Point differences generate $\operatorname{Aug}\mathbb{Z}[P_0]$. The pure shear is zero precisely when all $-W(s_\alpha)$ are equal; within each block they already are equal. The rational rank of the linear map consisting of their differences is $k$ for each of the $d$ independent torus coordinates, hence $dk$. The pure winding sublattice has finite index, so restriction does not change that rank. Its integer image is free abelian of this rank. A finite full image is equivalent to zero pure image by Corollary 7.2, which occurs precisely for $k = 0$.

For at least two punctures the augmentation permutation is faithful. Trivial action requires all congruences in (8.4) hold, by (8.1). Once pure, it acts trivially exactly under the second line of (8.4). Surjectivity of ambient winding then proves the image quotient and the triviality criterion. If there is a single puncture, irredundancy implies a single horizontal row with $c_\alpha = 1$; its slope differences are empty and its congruence is automatic, so the same conclusions hold. With no horizontal rows the action is identity and no reference block is chosen. □

The lattice in (8.4) is specified by integer congruences and rational linear equations. Clearing the denominators and introducing integer variables for the congruences gives an integer kernel, from which Smith normal form recovers all finite invariant factors and the free rank of the image. The finite puncture-permutation image is not in general its torsion subgroup: an infinite-order shear element can also have a nontrivial finite permutation.

Two arrangements illustrate the distinct behaviors. For rows $\{\beta_1, c\beta_0 + \beta_1\}$ over $\mathbb{T}^d \setminus \{0\}$ there are $c^d$ roots in one block and image $(\mathbb{Z}/c)^d$, with no pure shear. For rows $\{\beta_1, \beta_0, \beta_0 - \beta_1\}$, the

punctures are $\{0, b\}$, globally ordered. A primitive ambient loop of $b$ gives the relative arc action $e \mapsto e + W$. The image is infinite despite the ordered punctures. Both are essential irredundant bundle arrangements. In the noncompact version of the first example the same permutation acts with ambient identity; for the cylinder with $c = 2$, it swaps the two meridians and fixes the end-circle.

## 9. Sections of noncompact projections and compact-circle bundles

We return in this section to arbitrary kernel rank. The noncompact construction applies directly to a split projection of the ambient group.

**Proposition 9.1** (Escape in a real direction)**.** *Let $G = \mathbb{T}^d \times \mathbb{R}^v$, $v > 0$, and split $T = G^q \times G^k$. For any finite affine integer-wall arrangement, let $B$ be the complement of its vertical walls in $G^q$. Then the complement projection over $B$ admits a smooth section.*

*Proof.* Write a horizontal wall as

$$\lambda_L(b) + \lambda_K(y) = t_\lambda, \qquad \lambda_K \neq 0.$$

Choose $w \in \mathbb{R}^k$ outside the finitely many hyperplanes $\lambda_K(w) = 0$, and choose the first real coordinate of $G$. Let $r_\lambda(b)$ be that coordinate of $t_\lambda - \lambda_L(b)$, and put

$$R(b) = 1 + \sum_{\lambda \text{ horizontal}} \frac{\sqrt{1 + r_\lambda(b)^2}}{|\lambda_K(w)|}.$$

Define $y_j(b)$ to have torus coordinate zero, first real coordinate $R(b)w_j$, and all other real coordinates zero. This is smooth and globally defined. For each horizontal wall,

$$|R(b)\lambda_K(w)| > |r_\lambda(b)|,$$

so its real equation fails. All vertical equations fail by $b \in B$. Thus $(b, y(b))$ is the desired section. With no horizontal walls use $y = 0$; if $k = 0$ none can occur. □

This extends the known noncompact corank-one section phenomenon in [1, Corollary 3.1.2 and Remark 3.1.3] to arbitrary-rank affine split projections.

**Proposition 9.2** (The compact-circle boundary)**.** *For $G = \mathbb{T}$, every canonical arbitrary-rank affine $M$-ideal bundle in Theorem 1.1 has a smooth section.*

*Proof.* Its base arrangement is essential in $\mathbb{T}^q$. Lift its walls to $\mathbb{R}^q$. Every chamber is convex and bounded, since the normals span $(\mathbb{R}^q)^*$ and each row value is bounded between consecutive wall levels. A nonzero integer translation cannot stabilize a bounded chamber, and meeting translates are the same chamber. Projection therefore identifies it diffeomorphically with one base component: every path in that component lifts within the chamber. The component is thus contractible. Choose a point in the fiber over a chamber base point. Lifting the straight-line contraction, in reverse, through the fiber bundle gives a continuous section on that component. The homotopy lifting property used here is the usual one for a numerable bundle; bundles over manifolds are numerable.

Because the ambient projection is a global product, a section is a map $y : B \to \mathbb{T}^{n-q}$. At every base point its distance to the finite closed fiber-wall union is positive. A sufficiently fine smooth approximation to $y$, with continuous pointwise tolerance below this distance, still avoids every wall. The smooth approximation theorem for maps of manifolds [6, Chapter 2] therefore supplies a smooth section. Base components are open, so these maps combine to a global smooth section. For $q = 0$ the base is a point and the statement is immediate. □

The obstructions in the compact families below thus begin at real dimension two. The absence of an escaping real coordinate is substantive.

## 10. The compact isogeny family and its components

Recall the family (1.1). The map

$$\Phi_t(b, y) = (b_1, \dots, b_q, Cy + \sum_i B_i b_i - t) \tag{10.1}$$

is a torus covering of degree $|\det C|$, and $\mathcal{M} = \Phi_t^{-1}(P_*^{q+1})$. Locally choose a root $r(b)$ of $Cr(b) = t - \sum_i B_i b_i$. Translation by $-r(b)$ identifies the deleted set with $G[C]$. This directly proves local triviality with fiber $G \setminus G[C]$. The local roots suffice to construct these trivializations.

**Lemma 10.1.** *The horizontal deleted set $H_t = \{Cy + \sum_i B_i b_i = t\}$ has $[D : H]$ connected components. Each projects to $G^q$ as a covering of degree $|H|$ and meets the coordinate maximum $Y = \{b_1 = \cdots = b_q = 0\}$ in $|H|$ points. The coordinate-wall ideal is a corank-one $M$-ideal in the layer poset of this family.*

*Proof.* The projection $H_t \to G^q$ is a covering with root labels in $D$. Its ambient-loop monodromy is translation by $-\sum_i \beta_i W_i$, with image $H$. Its orbits are the cosets of $H$ in $D$. Connected components of a covering of a connected base correspond to these orbits, proving both component and degree counts.

Every horizontal component projects onto $G^q$, so it meets every coordinate layer; this verifies condition (1) for the coordinate ideal. Any partial wall intersection can be completed to a point with $b_i = 0$ and $Cy = t$. More explicitly, on any component of a partial intersection the projection to the remaining base coordinates has surjective derivative because $C$ is invertible over $\mathbb{R}$. It is a homomorphism of compact connected directions with connected target and is therefore onto. Thus setting those remaining coordinates to zero is possible. All maximal layers are the $|D|$ points just described. At each, the $q$ coordinate walls and one horizontal component have independent block equations. Their local interval is the Boolean lattice on $q + 1$ elements. The coordinate maximum is modular in this interval. The coordinate ideal is pure of rank $q$, an order ideal, and join-closed. This verifies condition (2) and proves the last assertion. □

The component computation uses only the block equations. In the holomorphic elliptic subfamily their local intervals are the corresponding complex hyperplane lattices.

10.1. **Free-group constraints on a section.** The punctured torus $P_*$ retracts to a rose with two circles. For instance put the puncture inside the unique top cell of a translated torus CW structure and radially retract that punctured cell to its attaching boundary. Its one-skeleton is a rose. Thus $P_*$ is a $K(F_2, 1)$, and its inclusion in $G$ induces the abelianization $\theta : F_2 \to \mathbb{Z}^2$. The base $B = P_*^q$ is a $K(F_2^q, 1)$.

**Lemma 10.2** (Commuting factor images)**.** *For a homomorphism $\varphi : F_2^q \to F_2$, either at most one factor has nontrivial image, or all factor images lie in one cyclic subgroup of $F_2$.*

*Proof.* A nontrivial element of a free group acts on its Cayley tree with a unique axis. An element centralizing it preserves that axis and its direction. Its translation length on the axis gives an injection of the centralizer into $\mathbb{Z}$: zero translation would fix a tree vertex, impossible for a nontrivial element in the free action. The centralizer is therefore infinite cyclic. Commuting nontrivial elements have the same axis and the same maximal cyclic centralizer. If two factor images are nontrivial, choose nontrivial $a, b$ from them. They commute. Every element of the first image commutes with $b$, every element of the second with $a$, and every remaining factor commutes with $a$. All these images consequently lie in the same cyclic centralizer. □

Suppose a continuous section is $y = f(b)$. Then

$$g(b) = Cf(b) + \sum_i B_i b_i - t$$

is a map $B \to P_*$. On $H_1$ of factor $i$, its matrix is

$$A_i = CF_i + B_i, \tag{10.2}$$

where $F_i$ is the integer $H_1$ matrix of $f$ on that factor. Therefore $A_i \bmod C\mathbb{Z}^2 = \beta_i$. Apply Lemma 10.2 to $g_* : F_2^q \to F_2$. If there is at most one nontrivial factor, at most one $\beta_i$ is nonzero. Otherwise all factor images lie in $\langle u \rangle$. For $v = \theta(u)$ every column of every $A_i$ is an integer multiple of $v$. All $\beta_i$ images thus lie in the cyclic subgroup generated by $v \bmod C$ in $D$, and their sum $H$ is cyclic. This proves necessity of (1.2).

10.2. **Realizing the congruences by maps.** If precisely one $\beta_j$ is nonzero, construct a map $P_* \to P_*$ with $H_1$ matrix $B_j$. On its rose send the two generators, column by column, to the words $x^a y^b$ for the columns $(a, b)$ of $B_j$, and compose with the retract and inclusion. Let $g$ depend only on this $j$th factor. Then $A_j = B_j$, $A_i = 0$ for $i \neq j$, and every column of $A_i - B_i$ lies in $C\mathbb{Z}^2$. If all $\beta_i$ vanish, take $g$ constant in $P_*$ instead.

If $H$ is nonzero cyclic, choose a generator $h$ and an integer lift $v \in \mathbb{Z}^2$. For each $i$ choose an integer row $\ell_i$ so that

$$B_i \equiv v\ell_i \pmod{C\mathbb{Z}^2}$$

column by column. Choose a smooth loop $\gamma : \mathbb{T} \to P_*$ with homology class $v$; a word in the rose provides such a loop after smoothing. Set

$$g(b) = \gamma\left(\sum_i \ell_i b_i\right).$$

Integer rows define the circle parameter on the original torus coordinates. This map has $A_i = v\ell_i$, so again every $A_i - B_i$ column lies in $C\mathbb{Z}^2$. The case $H = 0$ was already covered by the constant construction.

For either construction define

$$k(b) = g(b) - \sum_i B_i b_i + t : B \longrightarrow G.$$

The target fundamental group is $\mathbb{Z}^2$, and on every factor generator the image is a column of $A_i - B_i$. It is therefore contained in $C\mathbb{Z}^2$. The covering-lift criterion for $[C] : G \to G$ supplies $f : B \to G$ with $Cf = k$. The base is connected and locally path connected, and the covering is onto, so an initial lift can always be chosen. The resulting identity $Cf + \sum_i B_i b_i - t = g \neq 0$ proves that $f$ is a section. In the rose construction, approximate $g$ smoothly within the open manifold $P_*$, keeping its homotopy class and hence its $H_1$ matrices. Its lift is then smooth, since a covering is a local diffeomorphism. The cyclic construction is already smooth. This proves the continuous and smooth equivalence in Theorem 1.3.

10.3. **The fundamental-group extension.** The inverse image in (10.1) is connected. Indeed the last punctured-torus factor of $P_*^{q+1}$ already surjects on the deck quotient $D$, so its covering action is transitive. Its fundamental group is explicitly

$$\pi_1(\mathcal{M}) = \ker \kappa \leq F_2^q \times F_2, \qquad \kappa(x, z) = \theta(z) - \sum_i B_i \theta(x_i) \bmod C\mathbb{Z}^2. \tag{10.3}$$

This follows from the ambient covering lattice of (10.1); the phase translation does not change the lattice. The map $\kappa$ is onto, so the index is $|D|$. Projection to $F_2^q$ is onto, since $\theta(z)$ can represent any required element of $D$. Its kernel is $\ker(F_2 \xrightarrow{\theta} \mathbb{Z}^2 \to D)$, an index-$|D|$ subgroup of $F_2$ and a free group of rank $|D| + 1$. The rank follows directly from the covering of the two-circle rose: it has $|D|$ vertices and $2|D|$ edges. We obtain the exact sequence

$$1 \longrightarrow F_{|D|+1} \longrightarrow \pi_1(\mathcal{M}) \longrightarrow F_2^q \longrightarrow 1. \tag{10.4}$$

A splitting in (10.3) must be $x \mapsto (x, \varphi(x))$, where $\varphi : F_2^q \to F_2$ satisfies $\theta\varphi \equiv \sum_i B_i\theta_i \bmod C\mathbb{Z}^2$. Lemma 10.2 gives the same necessary condition as above. The constructed maps $g$ give the needed homomorphisms for sufficiency. Thus (10.4) splits exactly under (1.2). All spaces here are aspherical: the total space is a finite cover of $P_*^{q+1}$, and the fiber is a punctured torus.

10.4. **Finite covers admitting sections.** For each $i$ put

$$L_i = \{v \in \mathbb{Z}^2 : B_i v \in C\mathbb{Z}^2\}, \qquad G_i = \mathbb{R}^2/L_i.$$

Its index is $|\operatorname{im}\beta_i|$. The cover $G_i \to G$ restricts to a connected finite cover $P_i' \to P_*$ after deleting every preimage of zero. It is connected because a torus remains connected after finitely many points are removed. On it the formula

$$h_i([v]_{L_i}) = [C^{-1}B_i v]_{\mathbb{Z}^2}$$

is well defined and smooth, and $Ch_i = B_i b_i$. It requires no commutation of $C$ with $B_i$. If $q \geq 1$, choose one factor $j$ to leave uncovered. The single-factor construction above gives $h_j$ with $Ch_j + B_j b_j - t = g_j(b_j) \neq 0$. Cover every other factor by $P_i'$; then $f = h_j - \sum_{i\neq j} h_i$ is a section of the pullback. The covering degree is $\prod_{i\neq j} |\operatorname{im}\beta_i|$. For $q = 0$ the base is a point; for $q = 1$ the original bundle already has a section.

10.5. **All-parameter classification and finite lattice criteria.** If $q \geq 2$ and $D$ is cyclic, every subgroup $H$ is cyclic and every coefficient family has a section. If $D$ is noncyclic, choose $B_1 = B_2 = I$, with other $B_i = 0$. Then two $\beta_i$ are nonzero and $H = D$, so no section exists. Smith normal form has invariant factors $d_1 \mid d_2$, with $d_1$ the gcd of the four entries of $C$; $D$ is cyclic exactly when $d_1 = 1$. This proves the remaining assertion of Theorem 1.3.

For an individual family, $\beta_i = 0$ exactly when every entry of $C^{-1}B_i$ is integral. If more than one is nonzero, form the full lattice $L = C\mathbb{Z}^2 + \sum_i B_i\mathbb{Z}^2$ and choose an integer lattice basis $U$ with $L = U\mathbb{Z}^2$. Then $U^{-1}C$ is integer and $H \cong \mathbb{Z}^2/(U^{-1}C)\mathbb{Z}^2$. It is cyclic exactly when the gcd of the entries of $U^{-1}C$ is one. These finite integer conditions decide the section criterion.

**Corollary 10.3.** *Every bundle (1.1) has finite cohomological monodromy, with image isomorphic to $H$. This does not imply existence of a section.*

*Proof.* All roots move with the same lifted displacement $-C^{-1}\sum_i B_i W_i$. For a pure loop all their closed homology tracks are equal, so the relative shear vanishes by (7.5). The puncture-permutation image is $H$ and is realized by ambient windings, since $P_*^q \to G^q$ surjects on fundamental groups. For $|D| \geq 2$ its action on the augmentation group is faithful, so the full cohomology image is exactly $H$. For $|D| = 1$ both are trivial. The family with $C = cI$, $B_1 = B_2 = I$, $c \geq 2$, has $H = (\mathbb{Z}/c)^2$ noncyclic and no section by Theorem 1.3. □

## 11. Scalar isogenies in every compact dimension

The two-dimensional matrix theorem uses the free fundamental group of a punctured torus. Scalar isogenies admit a different argument, valid in every dimension at least two. It also explains why the obstruction to a section can persist when the fundamental-group extension splits.

Let $G = \mathbb{T}^d$, $d \geq 2$, $P = G \setminus \{0\}$, $c \geq 1$, and $a_1, \dots, a_q \in \mathbb{Z}$. For $t \in G$ define

$$\mathcal{M}_{c,a,t} = \left\{ (b, y) \in P^q \times G : cy + \sum_{i=1}^{q} a_i b_i \neq t \right\}.$$

Projection to $P^q$ is a smooth bundle: locally lift $t - \sum_i a_i b_i$ under $[c] : G \to G$ and translate the fiber $G \setminus G[c]$. Call $i$ active if $c \nmid a_i$.

**Theorem 11.1** (Complete scalar classification)**.** *The projection $\mathcal{M}_{c,a,t} \to P^q$ has a continuous section if and only if it has a smooth section, and this occurs exactly when at most one index is active. This statement holds for every phase $t$ and every $d \geq 2$.*

*Proof.* Choose the product CW structure on $\mathbb{T}^d$ with one cell in each coordinate dimension, translated so that the deleted point is in the interior of the unique $d$-cell. Radially retract the punctured closed top cell to its boundary, fixing that boundary. This gives a deformation retraction of $P$ onto the $(d-1)$-skeleton $K$. Thus $H^d(P;\mathbb{Q}) = 0$, whereas its integral homology in degrees 1 and $d-1$ identifies with that of $G$:

$$H_1(P;\mathbb{Z}) = \mathbb{Z}^d, \qquad H_{d-1}(P;\mathbb{Z}) = \bigwedge^{d-1} \mathbb{Z}^d.$$

These identifications hold also for $d = 2$, when the two groups coincide. They follow either from the cellular boundary, which is zero for the product torus cells, or from the relative homology sequence at the deleted point.

Suppose a section is $y = f(b)$, and set

$$g(b) = cf(b) + \sum_i a_i b_i - t \in P.$$

On the $i$th factor its map to $G$ has first-homology matrix $A_i = cF_i + a_i I$, with $F_i \in M_d(\mathbb{Z})$. If $i$ is active, then $A_i$ is invertible over $\mathbb{Q}$: singularity would give $F_i$ the rational eigenvalue $-a_i/c$. A rational root of a monic integer polynomial is integral, contradicting $c \nmid a_i$.

If two indices $i < j$ are active, the invertibility of $A_i$ and $A_j$ permits a choice of $x, y_2, \dots, y_d \in \mathbb{Q}^d$ such that $A_i x, A_j y_2, \dots, A_j y_d$ form a basis. Put $z = y_2 \wedge \cdots \wedge y_d$. Then

$$A_i x \wedge (\bigwedge^{d-1} A_j) z \neq 0 \quad \text{in} \quad \bigwedge^{d} \mathbb{Q}^d.$$

Realize $x$ and $z$ as rational homology classes in the $i$th and $j$th factors of $P^q$, fixing points in the others. If $\eta_1, \dots, \eta_d$ are the coordinate classes of $G$, their orientation product pulls back on these two factors as

$$\bigwedge_{r=1}^{d} (\mathrm{pr}_i^* A_i^* \eta_r + \mathrm{pr}_j^* A_j^* \eta_r).$$

Its bidegree-$(1, d-1)$ component evaluates on $x \times z$ as $\det(A_i x, A_j y_2, \dots, A_j y_d)$, which is nonzero. On the other hand, $g$ factors through $P$ and $H^d(P;\mathbb{Q}) = 0$, so the same pullback must vanish. This proves necessity.

For sufficiency first construct, for every integer $a$, a map $\gamma : P \to P$ inducing $aI$ on $H_1$. The multiplication map $[a] : G \to G$ has a cellular approximation for the chosen CW structure, so it maps $K$ into $K$. Compose the retraction $P \to K$, this cellular map, and the inclusion $K \to P$. The resulting $\gamma$ has the required first-homology map. Approximate it by a smooth map within the open manifold $P$, preserving its homotopy class.

If there is one active index $j$, use $a = a_j$. The map

$$u(b_j) = \gamma(b_j) - a_j b_j + t : P \longrightarrow G$$

induces zero on first homology, hence on fundamental groups, because the target fundamental group is abelian. It therefore lifts through the covering $[c] : G \to G$ to a smooth map $h$ with $ch = u$. Then

$$f(b) = h(b_j) - \sum_{i \neq j} (a_i/c) b_i$$

is well defined, and $cf(b) + \sum_i a_i b_i - t = \gamma(b_j) \neq 0$. If no index is active, choose a constant $y_0$ with $cy_0 - t \neq 0$ and put $f(b) = y_0 - \sum_i (a_i/c) b_i$. This also covers the case $q = 0$. Both constructions are smooth and prove sufficiency. □

For $d = 2$, Theorem 11.1 agrees exactly with the matrix criterion. Here $D = (\mathbb{Z}/c)^2$ and, writing $g_0 = \gcd(c, a_1, \dots, a_q)$, one has $H \cong (\mathbb{Z}/(c/g_0))^2$. This group is cyclic only when it is trivial. If at least two indices are active, the other alternative in (1.2) also fails.

The full finite kernel is essential. For example $c = 6$, $a_1 = 2$, $a_2 = 3$ gives no section. Each of the primes 2 and 3 separately sees only one active factor, but the full coefficient family has two. By contrast, for an elliptic curve with complex multiplication by $i$, the endomorphism $2 + i$ has lattice matrix

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \qquad D \cong \mathbb{Z}/5.$$

Every matrix coefficient family for this isogeny has a section. The scalar isogeny $2I$ has kernel quotient $(\mathbb{Z}/2)^2$ and fails already for two identity coefficients.

11.1. **A three-wall obstruction and finite covers.** Consider the three connected walls in $G^3$

$$b_1 = 0, \qquad b_2 = 0, \qquad b_1 + b_2 + cy = 0, \quad c \geq 2. \tag{11.1}$$

Their normals are linearly independent. Every single wall and every pairwise intersection is connected; the triple intersection consists of the $c^d$ points with $b_1 = b_2 = 0$ and $cy = 0$. The ideal generated by the first two walls is an $M$-ideal of corank one. To check the global condition, the third wall projects onto the entire $(b_1, b_2)$-torus and hence meets every maximal vertical layer. In each interval below a triple point the three independent normals give a Boolean lattice, in which the rank-two vertical element is modular. The join and order-ideal conditions follow from these intersections. Thus this is a bundle in the Bibby–Delucchi class, with base $P^2$ and fiber $G \setminus G[c]$.

For every $d \geq 2$ it has no section by Theorem 11.1. Nevertheless a concrete finite cover restores one. Pull back along

$$(G \setminus G[c]) \times P \longrightarrow P^2, \qquad (x, b_2) \longmapsto (cx, b_2).$$

This is a cover of degree $c^d$, and $y = -x$ is a section because $cx + b_2 + c(-x) = b_2 \neq 0$. More generally, covering all but one factor by $G \setminus G[c] \to P$ gives a section for every scalar family, of degree $c^{d(q-1)}$ when $q \geq 1$. The last factor is handled by the single-factor construction in the proof. These are finite covers of complements.

For $d = 1$ the same three walls do have a section. On the unique lifted chamber $0 < b_1, b_2 < 1$ of the base, set

$$y = (1/2 - b_1 - b_2)/c \pmod{\mathbb{Z}}.$$

The horizontal expression is the constant $1/2$. This also illustrates why the circle argument in Proposition 9.2 has a different conclusion.

For $d = 2$, the complement of (11.1) is a connected degree-$c^2$ cover of $P^3$; its fundamental-group extension does not split, by Theorem 1.3. For $d \geq 3$, the inclusion $P \to G$ induces an isomorphism on fundamental groups. This follows from the above $(d-1)$-skeleton description: adding the top cell, of dimension at least three, does not change $\pi_1$. The same covering description identifies its fundamental group with the lattice

$$\left\{ (v_1, v_2, v_3) \in (\mathbb{Z}^d)^3 : v_3 - v_1 - v_2 \in c\mathbb{Z}^d \right\}.$$

Projection to the first two factors has the group section $(v_1, v_2) \mapsto (v_1, v_2, v_1 + v_2)$, whereas the bundle has no continuous section. In higher dimension, top cohomology records the obstruction to a continuous section even when the fundamental-group extension splits.

## 12. Connected intersections and the section criterion

In the matrix family, each connected horizontal component meets $Y = \{b_1 = \cdots = b_q = 0\}$ in $|H|$ points, by Lemma 10.1. Thus the connected-intersection hypothesis in [1, Remark 3.1.3] is equivalent here to $H = 0$. Theorem 1.3 also gives sections for nonzero cyclic $H$ and for a single nonzero coefficient map $\beta_i$. The criterion depends on the subgroup generated by the full coefficient family in the finite isogeny kernel.

Corollary 10.3 and the three-wall example show how monodromy and sections differ in this family. Its cohomological monodromy is finite, while the section obstruction can be nonzero. Covering the

appropriate punctured base factors kills their coefficient maps in $D$ and gives the explicit sections constructed above. For $d \geq 3$ the scalar obstruction is detected by a mixed top-cohomology product even though the fundamental-group extension splits.

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