INTEGRAL COHOMOLOGY OF CYCLIC COVERS UNDER THE CDO CONDITION

XIE'S CONJECTURE 1.9 FOR ARBITRARY COMPLEX ARRANGEMENTS

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Abstract. We prove that every connected finite cyclic cover of an essential complex hyperplane arrangement complement satisfying the cyclic CDO condition has torsion-free integral cohomology. This answers Conjecture 1.9 of Xie, with the same ranks as in his theorem for complexified real arrangements. The proof establishes a valuation-one nonresonance theorem over an arbitrary discrete valuation ring. A length comparison in the Wang sequence excludes extensions of residue-field torsion to thicker torsion. The resulting cochain decomposition determines the entire deck representation over every field, including characteristics dividing the covering degree. An explicit lattice splitting gives the corresponding p-adic and finite-coefficient representations. As a further consequence, the CDO condition implies Aomoto vanishing over every field for arbitrary complex arrangements.

1. The question and the cyclic-cover theorem

Let $\mathcal{A}$ be a finite essential arrangement of affine complex hyperplanes in $\mathbb{C}^\ell$, where $\ell \geq 1$, and let

$$M = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H.$$

Essential means that the linear parts of the defining equations span the dual of $\mathbb{C}^\ell$. Fix its projective compactification and write $\overline{\mathcal{A}} = \{\overline{H} : H \in \mathcal{A}\} \cup \{H_\infty\}$. An *edge* is a nonempty intersection of members of $\overline{\mathcal{A}}$. An edge X is *dense* when the central arrangement normal to X formed by the hyperplanes containing X is indecomposable. In particular, a single hyperplane is a dense edge.

Choose an epimorphism

$$\varphi : H_1(M;\mathbb{Z}) \longrightarrow C_N = \mathbb{Z}/N\mathbb{Z}$$

and let $Y \to M$ be the associated connected cover. Meridians about the affine hyperplanes form a basis of $H_1(M;\mathbb{Z})$; put $\epsilon_H = \varphi(\gamma_H)$ and $\epsilon_\infty = -\sum_H \epsilon_H$. For an edge define

$$\epsilon_X = \sum_{H \in \overline{\mathcal{A}},\ H \supseteq X} \epsilon_H \in C_N.$$

Definition 1.1. The cover satisfies the *cyclic CDO condition along* H_∞ if ϵ_X generates C_N for every dense edge $X \subseteq H_\infty$. Equivalently, $\gcd(\widetilde{\epsilon}_X, N) = 1$ for any integer lift $\widetilde{\epsilon}_X$ of ϵ_X.

Put

$$b_i = \operatorname{rank} H^i(M;\mathbb{Z}), \qquad \beta_i = \sum_{j=0}^{i} (-1)^{i-j} b_j, \qquad \beta_{-1} = 0. \tag{1.1}$$

Thus $b_i = \beta_{i-1} + \beta_i$ and $\beta_\ell = (-1)^\ell \chi(M)$.

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Xie proves integral torsion-freeness under Definition 1.1 for complexified real arrangements and asks, in Conjecture 1.9 of [7], whether this extends to all essential complex arrangements. The following theorem gives the extension, together with its additive deck-module consequences.

Theorem 1.2. *Suppose that $Y \to M$ satisfies the cyclic CDO condition. Then*

$$H^i(Y;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{b_i}, & 0 \le i < \ell, \\ \mathbb{Z}^{b_\ell + (N-1)\beta_\ell}, & i = \ell, \\ 0, & i > \ell. \end{cases} \tag{1.2}$$

For every field k, as $k[C_N]$-modules,

$$H^i(Y;k) \cong \begin{cases} k^{b_i}, & 0 \le i < \ell, \\ k^{\beta_{\ell-1}} \oplus k[C_N]^{\beta_\ell}, & i = \ell, \\ 0, & i > \ell. \end{cases} \tag{1.3}$$

All copies of k displayed separately have trivial action. Moreover, (1.3) holds with k replaced by $\mathbb{Z}_p$ for every prime p and by $\mathbb{Z}/m\mathbb{Z}$ for every positive integer m. The same formulas hold for homology with the indicated coefficients. All these isomorphisms are isomorphisms of modules; they need not be canonical.

The main step is a theorem for a local system over a discrete valuation ring. It controls torsion extensions before passing to a finite group ring, where semisimplicity may fail. Sections 2–4 prove that theorem; Sections 5–6 prove Theorem 1.2. Section 7 gives the Aomoto consequence.

2. Valuation-one nonresonance

Let $\mathcal{O}$ be a discrete valuation ring with uniformizer s, residue field k, fraction field K, and normalized valuation v_s. A rank-one local system of free $\mathcal{O}$-modules on M is determined by units q_H assigned to meridians. Set

$$q_\infty = \Big(\prod_{H \in \mathcal{A}} q_H\Big)^{-1}, \qquad q_X = \prod_{H \in \overline{\mathcal{A}},\ H \supseteq X} q_H. \tag{2.1}$$

We require

$$q_H \in 1 + s\mathcal{O} \quad (H \in \mathcal{A}), \qquad v_s(q_X - 1) = 1 \quad \text{for every dense } X \subseteq H_\infty. \tag{2.2}$$

The reduction of the local system modulo s is therefore constant. No completeness assumption is imposed on $\mathcal{O}$.

Theorem 2.1 (DVR nonresonance)**.** *Let $\mathcal{L}$ satisfy (2.2). Then*

$$H^i(M;\mathcal{L}) \cong \begin{cases} k^{\beta_{i-1}}, & 1 \le i < \ell, \\ \mathcal{O}^{\beta_\ell} \oplus k^{\beta_{\ell-1}}, & i = \ell, \\ 0, & i = 0 \text{ or } i > \ell. \end{cases} \tag{2.3}$$

In particular, every torsion summand is annihilated by s.

We use the finite CW type of complex arrangement complements in dimensions at most ℓ, their torsion-free integral cohomology generated by meridian classes, and the integral deletion–restriction residue epimorphism. The latter remains an epimorphism after base change to any field. These facts follow from the Orlik–Solomon description [5]. Minimality [3] will be used only after Theorem 2.1 has been proved.

The induction follows the generic-pencil construction of Liu–Maxim–Wang [4]. Their generic-projection theorem applies over a Dedekind domain, hence over $\mathcal{O}$. We use both the standard and the dual perverse t-structures in this comparison. Lengths over $\mathcal{O}$ replace the 2-primary cardinalities in their argument.

2.1. **A generic pencil and its eigenvalues.** Choose a generic affine hyperplane H_0 with defining function l_0 and put

$$M' = M \setminus H_0, \qquad M'' = M \cap H_0, \qquad f = l_0 : M' \longrightarrow \mathbb{C}^*.$$

In projective coordinates this is the pencil l_0/l_∞. Let Δ_∞^* be a sufficiently small punctured disc about infinity and put $U = f^{-1}(\Delta_\infty^*)$. The map $U \to \Delta_\infty^*$ is a bundle. Its fibre F is a generic affine section of M of dimension $\ell - 1$. Both F and M'' inherit (2.2): the dense-edge products are the same products as those in the ambient arrangement; this is the geometric observation in [4, Lemma 2.1]. The system $\mathcal{L}$ extends across H_0. Write T for the bundle monodromy on $H^*(F;\mathcal{L})$, including transport of coefficients.

Lemma 2.2. *Every eigenvalue α of T on $H^*(F;\mathcal{L}) \otimes_{\mathcal{O}} K$ is q_X or q_X^{-1} for a dense edge $X \subseteq H_\infty$. In particular, $v_s(\alpha - 1) = 1$.*

Proof. Resolve the pencil by blowing up $H_0 \cap H_\infty$ in $\mathbb{P}^\ell$. Nearby cycles commute with the proper direct image of this resolution. It suffices to inspect the local nearby-cycle stalks, as in [4, Proposition 2.6]. An annihilating polynomial for their monodromy, raised to a power, annihilates the hypercohomology; hence passage from local stalks to global cohomology adds no eigenvalues.

Away from the exceptional divisor the germ is a central arrangement with a distinguished hyperplane L_0 at infinity. Decompose it into indecomposable central factors and let the first factor contain L_0. Its total monodromy q_1 is q_X for a dense infinity edge, so $v_s(q_1 - 1) = 1$. If another factor has total monodromy $q_j \neq 1$, its central $\mathbb{C}^*$ factor has acyclic K-cochains, since $q_j - 1$ is invertible in K. That stalk contributes no eigenvalue. If all these other total monodromies equal one, the only coefficient multiplier of the scaling loop is q_1. Scaling all central coordinates once gives this scalar on the local fibre cohomology: the lifted loop is the sum of meridians. Reversing the loop replaces it by q_1^{-1}.

On the exceptional divisor the local nearby fibre is a central complement obtained from a generic section. Its factor containing the inherited hyperplane at infinity has total monodromy q with $v_s(q - 1) = 1$. Its central $\mathbb{C}^*$ factor is thus acyclic over K, so these stalks contribute no eigenvalues. These are precisely the two local models in the blow-up description of [4, Section 2]. Finally, $q^{-1} - 1 = -q^{-1}(q - 1)$ preserves the valuation. The argument uses eigenvalues with algebraic multiplicity and allows Jordan blocks. □

3. The length comparison at infinity

Lemma 3.1. *Let V be a finitely generated $\mathcal{O}$-module whose torsion is annihilated by s. Write $a = \dim_k V_{\mathrm{tor}}$ and $r = \operatorname{rank}_{\mathcal{O}} V$. Suppose an automorphism T has $v_s(\alpha - 1) = 1$ for every eigenvalue on $V \otimes K$. Then*

$$\operatorname{length}_{\mathcal{O}} \ker(T - 1) \leq a, \qquad \operatorname{length}_{\mathcal{O}} \operatorname{coker}(T - 1) \leq r + a.$$

Proof. The torsion submodule is T-invariant. On the free quotient V/V_{tor}, the map $T - 1$ is injective and its determinant has valuation r, by multiplying its eigenvalues minus one with algebraic multiplicity. Its cokernel has length r. The snake lemma gives

$$\ker(T - 1 \mid V) = \ker(T - 1 \mid V_{\mathrm{tor}}),$$

and

$$0 \longrightarrow \operatorname{coker}(T - 1 \mid V_{\mathrm{tor}}) \longrightarrow \operatorname{coker}(T - 1 \mid V) \longrightarrow \operatorname{coker}(T - 1 \mid V/V_{\mathrm{tor}}) \longrightarrow 0.$$

The kernel and cokernel on V_{tor} each have length at most a, proving both bounds. □

Assume inductively that Theorem 2.1 holds for F. Write

$$H^j(F;\mathcal{L}) \cong \mathcal{O}^{r_j} \oplus k^{a_j}, \qquad A_i = r_{i-1} + a_{i-1} + a_i,$$

with all indices outside the cohomological range interpreted as zero. Let $U_i = H^i(U;\mathcal{L})$. The Wang sequence gives

$$0 \longrightarrow \operatorname{coker}(T-1 \mid H^{i-1}(F;\mathcal{L})) \longrightarrow U_i \longrightarrow \ker(T-1 \mid H^i(F;\mathcal{L})) \longrightarrow 0. \tag{3.1}$$

Lemmas 2.2 and 3.1 imply that U_i has finite length and

$$\operatorname{length}_{\mathcal{O}} U_i \le A_i. \tag{3.2}$$

The ordinary monodromy of the pencil acts trivially on $H^*(F;k)$. Each hyperplane of the generic section comes from an ambient hyperplane, so its meridian class is the restriction of a global class on M'. These classes generate $H^*(F;k)$ and are fixed by transport. The ordinary Wang sequence gives

$$\dim_k H^i(U;k) = b_{i-1}(F) + b_i(F).$$

The universal coefficient sequence on F gives $b_j(F) = r_j + a_j + a_{j+1}$. Consequently

$$\dim_k H^i(U;k) = A_i + A_{i+1}. \tag{3.3}$$

For a finite-length $\mathcal{O}$-module V put $g(V) = \dim_k(V/sV)$. The structure theorem over a DVR shows that

$$\dim_k \operatorname{Tor}_1^{\mathcal{O}}(V,k) = g(V) \le \operatorname{length}_{\mathcal{O}} V,$$

and equality holds precisely when $sV = 0$. The space U has finite CW type: retracting the punctured base disc onto a circle identifies it up to homotopy with the mapping torus of F, which can be replaced by a finite CW model with cellular monodromy. The resulting local-system cochains are finite free over $\mathcal{O}$. Their universal coefficient sequence, followed by (3.3), yields

$$g(U_i) + g(U_{i+1}) = A_i + A_{i+1}, \qquad g(U_j) \le \operatorname{length}_{\mathcal{O}} U_j \le A_j. \tag{3.4}$$

Since both summands are bounded separately, equality of their sum forces $g(U_j) = \operatorname{length}_{\mathcal{O}} U_j = A_j$ in every degree. Thus

$$sH^i(U;\mathcal{L}) = 0 \quad \text{for every } i. \tag{3.5}$$

In particular, the Wang extensions have no cyclic summand $\mathcal{O}/(s^u)$ with $u > 1$.

4. From infinity to the whole complement

The following comparison carries the calculation at infinity into the deletion–restriction induction.

Lemma 4.1 (Generic-projection comparison)**.** *For the above generic pencil, if $H^*(U;\mathcal{L})$ is $\mathcal{O}$-torsion, then*

$$H^i(M';\mathcal{L}) \cong H^i(U;\mathcal{L}) \quad (i < \ell),$$

and restriction $H^\ell(M';\mathcal{L}) \to H^\ell(U;\mathcal{L})$ is injective on the torsion submodule.

Proof. We give the Dedekind-coefficient argument of [4, Proposition 4.1 and Theorem 4.2]. On $\mathbb{C}^*$ put $P_f = Rf_*(\mathcal{L}[\ell])$, and let $j_\infty : \mathbb{C}^* \hookrightarrow \mathbb{P}^1 \setminus \{0\}$. Put $A = R\Gamma(\mathbb{P}^1 \setminus \{0\}, j_{\infty!}P_f)$. The attaching triangle gives

$$A \longrightarrow R\Gamma(M';\mathcal{L}[\ell]) \longrightarrow R\Gamma(U;\mathcal{L}[\ell]) \longrightarrow A[1].$$

Write p for the standard perverse t-structure and p^+ for its Verdier dual. A constructible complex is strongly perverse when it lies in ${}^pD^{\le 0} \cap {}^{p^+}D^{\ge 0}$. The shifted free local system $\mathcal{L}[\ell]$ has this property. Write $j : M \hookrightarrow \mathbb{C}^\ell$ for the affine, quasi-finite arrangement inclusion and $g = l_0 : \mathbb{C}^\ell \to \mathbb{C}$ for the chosen generic linear projection. The complex $Rg_*Rj_*(\mathcal{L}[\ell])$ on $\mathbb{C}$ restricts to P_f on $\mathbb{C}^*$. Direct image by the affine, quasi-finite map j preserves the standard and dual perverse t-structures, so $Rj_*(\mathcal{L}[\ell])$ is strongly perverse. The generic-direct-image theorem [4, Theorem 4.2] therefore makes every ${}^{p^+}H^r(P_f)$ with $r \neq 0$ a shifted constant sheaf. These pieces contribute zero to A: the attaching triangle for a constant sheaf on $\mathbb{C}^*$ has an isomorphism from its global cohomology to its nearby cohomology at infinity. Set $Q_f = {}^{p^+}H^0(P_f)$ and $j_0 : \mathbb{C}^* \hookrightarrow C_0 = \mathbb{P}^1 \setminus \{\infty\}$. The perverse

hypercohomology spectral sequence and the exchange of the two complementary open embeddings give, in every degree,

$$H^r(A) \cong \mathbb{H}^r(\mathbb{P}^1 \setminus \{0\}, j_{\infty!}Q_f) \cong \mathbb{H}^r_c(C_0, Rj_{0*}Q_f).$$

Here $Rj_{0*}Q_f$ is dual-perverse because j_0 is affine and quasi-finite. Dual-perverse Artin vanishing gives

$$H^j(A) = 0 \ (j < 0), \qquad H^0(A) \text{ is } \mathcal{O}\text{-torsion-free.}$$

These consequences of dual-perverse Artin vanishing over a Dedekind domain are the steps in [4, formulas (12)–(19)].

The triangle gives the asserted isomorphisms in degrees below $\ell - 1$. At degree $\ell - 1$, the map from the torsion module $H^{\ell-1}(U; \mathcal{L})$ to the torsion-free $H^0(A)$ is zero, giving that isomorphism as well. It also makes $H^0(A) \to H^\ell(M'; \mathcal{L})$ injective. Hence the kernel of the top restriction is itself torsion-free, proving injectivity on torsion. □

Proof of Theorem 2.1. For $\ell = 1$, the complement retracts onto a rose with one loop per hyperplane. Its local-system coboundary is the column with entries $q_H - 1$. Every entry is divisible by s. If all were divisible by s^2, their product inverse would also be 1 modulo s^2, contrary to $v_s(q_\infty - 1) = 1$. Thus the sole nonzero Smith factor is s, which proves (2.3). A zero-dimensional section is a point with constant coefficients and provides the trivial endpoint.

Suppose $\ell \geq 2$. Apply the induction to F and M''. Equation (3.5) and Lemma 4.1 show that all cohomology of M' below degree ℓ and its torsion in degree ℓ are annihilated by s. Free cohomology occurs only in degree ℓ; above that degree the CW dimension gives vanishing.

Since $\mathcal{L}$ extends across H_0, deletion–restriction gives

$$\cdots \longrightarrow H^{i-2}(M''; \mathcal{L}) \longrightarrow H^i(M; \mathcal{L}) \longrightarrow H^i(M'; \mathcal{L}) \xrightarrow{\delta_i} H^{i-1}(M''; \mathcal{L}) \longrightarrow \cdots . \tag{4.1}$$

The map δ_ℓ is surjective because $H^{\ell+1}(M; \mathcal{L}) = 0$. For $1 \leq i < \ell$, the natural universal coefficient epimorphisms form the commuting square

$$\begin{array}{ccc} H^{i-1}(M'; k) & \longrightarrow & \operatorname{Tor}_1^{\mathcal{O}}(H^i(M'; \mathcal{L}), k) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \operatorname{Tor}_1(\delta_i, k)} \\ H^{i-2}(M''; k) & \longrightarrow & \operatorname{Tor}_1^{\mathcal{O}}(H^{i-1}(M''; \mathcal{L}), k). \end{array}$$

The left vertical map is the ordinary arrangement residue, hence surjective over every field. The right map is therefore surjective. Both modules to which $\operatorname{Tor}_1^{\mathcal{O}}(-, k)$ is applied are annihilated by s; the resolution $0 \to \mathcal{O} \xrightarrow{s} \mathcal{O} \to k \to 0$ identifies this functor naturally with those modules. Thus δ_i itself is surjective. The negative-degree terms are zero. Surjectivity of δ_{i-1} makes the preceding Gysin map $H^{i-2}(M''; \mathcal{L}) \to H^i(M; \mathcal{L})$ zero. Hence $H^i(M; \mathcal{L})$ embeds in $H^i(M'; \mathcal{L})$ for every i. Its torsion is annihilated by s, and its free part occurs only in degree ℓ.

The edge H_∞ itself satisfies (2.2); thus some meridian has nonidentity monodromy, and $H^0(M; \mathcal{L}) = 0$. Euler characteristic over K gives free rank β_ℓ at the top. If a_i is the number of residue-field torsion summands and r_i is the free rank, universal coefficients give

$$b_i = r_i + a_i + a_{i+1}.$$

Starting with $a_0 = r_0 = 0$ yields $a_i = \beta_{i-1}$ for $1 \leq i \leq \ell$. This proves the theorem. □

5. Primary cyclic blocks and integral torsion-freeness

We will use a simple fact about complexes over a DVR.

Lemma 5.1. *A bounded complex of finite free $\mathcal{O}$-modules is isomorphic to a direct sum of isolated copies of $\mathcal{O}$ and two-term complexes $\mathcal{O} \xrightarrow{s^u} \mathcal{O}$, where $u \geq 0$. If its reduction modulo s has zero differential, no $u = 0$ block occurs. If all its cohomological torsion is killed by s, no $u > 1$ block occurs.*

Proof. Diagonalize a nonzero differential by Smith operations. If a diagonal entry is nonzero, the equation $d^2 = 0$ forces the incoming component in its source coordinate and the outgoing component from its target coordinate to vanish, since $\mathcal{O}$ is a domain. This splits off that two-term complex. Repeat with the remaining coordinates. The last two assertions follow by reduction and by taking its cokernel. □

Choose integer lifts m_H of the affine weights and set $m_\infty = -\sum_H m_H$. Let k be algebraically closed of characteristic $p > 0$ and write

$$N = p^a d, \quad (p, d) = 1, \quad m = p^a.$$

The group algebra decomposes as

$$k[C_N] = k[t]/(t^N - 1) \cong \prod_{\zeta^d = 1} k[t]/((t - \zeta)^m). \tag{5.1}$$

The cochains of Y are the cochains of M with its regular cyclic local system, so this is also a decomposition of its cochain complex.

5.1. **The block at one.** Suppose first $p \mid N$. Set $\mathcal{O} = k[[s]]$ and $q_H = (1 + s)^{m_H}$. For each dense infinity edge,

$$q_X = 1 + m_X s + O(s^2), \qquad m_X \not\equiv 0 \pmod{p},$$

by the cyclic CDO condition. Theorem 2.1 applies. A minimal CW model for M has b_i cells in degree i and zero ordinary cellular coboundary [3]. Its completed local-system cochain complex C^* is free over $\mathcal{O}$ and every differential is in (s). By Lemma 5.1 and Theorem 2.1, its only nonzero two-term blocks are

$$\mathcal{O} \xrightarrow{s} \mathcal{O},$$

with β_{i-1} blocks targeting degree i, and its isolated free terms are β_ℓ copies of $\mathcal{O}$ in degree ℓ.

The substitution $t = 1 + s$ identifies the block of (5.1) at one with $\mathcal{O}/(s^m)$. After this reduction, each two-term block has kernel $s^{m-1}\mathcal{O}/(s^m)$ and cokernel $\mathcal{O}/(s)$; both are trivial one-dimensional deck modules. Each isolated top term gives a full-length block $\mathcal{O}/(s^m)$. Thus the principal block contributes

$$k^{b_i} \quad (i < \ell), \qquad k^{\beta_{\ell-1}} \oplus (\mathcal{O}/(s^m))^{\beta_\ell} \quad (i = \ell). \tag{5.2}$$

If $p \nmid N$, then $m = 1$ and the principal block is the ordinary constant local system, which gives the same formula directly. No valuation-one condition modulo such a prime is needed.

5.2. **The nonprincipal blocks.** For $\zeta \neq 1$ put $t = \zeta(1 + s)$ and use the completed complex over $\mathcal{O} = k[[s]]$. Its residue character has meridian values ζ^{m_H}. Its dense-edge values are nonidentity because the order of ζ divides N and $\gcd(m_X, N) = 1$. Field-valued CDO nonresonance therefore gives residue cohomology only in degree ℓ, of dimension β_ℓ.

The clean-extension proof of field nonresonance applies in every characteristic. Resolve the divisor at infinity along dense edges. Nonidentity scalar monodromy makes the punctured normal-direction cochain complex acyclic, since the scalar minus one is invertible in the field. The lower and upper extensions of the local system agree at infinity. Artin vanishing for field-coefficient perverse sheaves then gives vanishing below ℓ, while CW dimension gives vanishing above ℓ. Euler characteristic gives the stated dimension. This is the clean-extension argument of CDO [2], with the coefficient field left arbitrary.

Apply Lemma 5.1 to this completed complex. A two-term s^u block with $u \geq 1$ reduces to a zero map and would contribute one residue class in each of two consecutive degrees. Since residue cohomology is concentrated in degree ℓ, such a block cannot occur. The same concentration excludes isolated terms in every other degree. The two-term blocks are therefore contractible, and the only isolated terms are β_ℓ copies of $\mathcal{O}$ at the top. Reduction modulo s^m gives β_ℓ full-length blocks at ζ and no other cohomology.

Combining all factors proves (1.3) over algebraically closed fields of positive characteristic. In characteristic zero the ordinary field CDO theorem gives the same statement by semisimplicity. For an arbitrary field, scalar extension commutes with the cohomology of the finite local-system complex. Since t^d-1 is separable, each of its irreducible factors splits into distinct roots over an algebraic closure. The elementary-divisor classification of finite $k[t]$-modules therefore descends the result: every irreducible factor of t^d-1 has only its maximal exponent m, with multiplicity β_ℓ, apart from the extra simple $t-1$ factors. These are exactly the regular and trivial modules in (1.3).

Proof of (1.2). Over $\mathbb{Q}$, (1.3) gives the ranks in (1.2). Over every $\mathbb{F}_p$ it gives the same dimensions. If $t_i(p)$ denotes the number of p-primary cyclic summands of $H^i(Y;\mathbb{Z})$, the integral universal coefficient sequence gives

$$\dim_{\mathbb{F}_p} H^i(Y;\mathbb{F}_p) = \operatorname{rank} H^i(Y;\mathbb{Z}) + t_i(p) + t_{i+1}(p).$$

Every summand on the right beyond the rank is nonnegative, so all $t_i(p)$ vanish, for every p and every i. This proves (1.2); integral homology is also torsion-free by universal coefficients. The case $N=1$ is the ordinary arrangement cohomology theorem. □

6. Lifting the deck representation

The field representations lift to local integral lattices by the following equivariant splitting.

Lemma 6.1. *Let $\mathcal{O}$ be a DVR with residue field k and fraction field K. Let G be finite with* $\operatorname{char} K \nmid |G|$. *Suppose V is a finite free $\mathcal{O}$-module with a G-action and*

$$V/sV \cong k[G]^c \oplus k^a, \qquad V \otimes_{\mathcal{O}} K \cong K[G]^c \oplus K^a.$$

Then $V \cong \mathcal{O}[G]^c \oplus \mathcal{O}^a$ as $\mathcal{O}[G]$-modules, with trivial action on the last summand.

Proof. Lift the standard generators of the c regular residue summands to $v_1,\dots,v_c \in V$, obtaining a map $E : \mathcal{O}[G]^c \to V$. Its reduction has full column rank. Thus a maximal minor of E is a unit, and all its Smith factors are units. It follows that E is injective and has free cokernel; thus it splits as a map of $\mathcal{O}$-modules.

For each j, extend the function taking the coefficient of 1_G in the jth coordinate of $\mathcal{O}[G]^c$ along E to a linear function $\lambda_j : V \to \mathcal{O}$. Define

$$\Pi_j(v) = \sum_{g\in G} \lambda_j(g^{-1}v)g.$$

Changing the summation index gives $\Pi_j(hv) = h\Pi_j(v)$. If $a \in \mathcal{O}[G]^c$, then $\lambda_j(g^{-1}E(a))$ is the coefficient of g in a_j. Consequently $\Pi_j E(a) = a_j$. Hence $\Pi E = \mathrm{id}$ and $V = E(\mathcal{O}[G]^c) \oplus W$ as group-ring modules. Maschke's theorem over K cancels the already split regular part, leaving $W \otimes K$ trivial of dimension a. Since W is torsion-free, every $g-1$ already vanishes on W. Thus $W \cong \mathcal{O}^a$ with trivial action. □

Apply Lemma 6.1 to $V = H^\ell(Y;\mathbb{Z}_p)$ and $G = C_N$. Integral torsion-freeness identifies its reduction with $H^\ell(Y;\mathbb{F}_p)$, and the characteristic-zero representation gives its $\mathbb{Q}_p$ extension. This proves the $\mathbb{Z}_p$ formula in Theorem 1.2. Below the top degree the action is already trivial over $\mathbb{Q}_p$, so it is trivial on the free $\mathbb{Z}_p$-lattice as well. Reduction modulo p^r and the Chinese remainder theorem give the finite-coefficient formula. Duality changes a deck generator to its inverse; regular and trivial modules are self-dual, giving the homology formulas. This finishes the proof of Theorem 1.2.

Corollary 6.2. *Over an algebraically closed field of characteristic $p>0$, write $N = p^a d$ with $(p,d)=1$. Below degree ℓ the deck generator has b_i blocks $J_1(1)$. In degree ℓ it has $\beta_{\ell-1}$ additional blocks $J_1(1)$ and β_ℓ blocks $J_{p^a}(\zeta)$ for every dth root ζ.*

Remark 6.3. The integral formula (1.2) describes the underlying free abelian groups. The equivariant regular–trivial splittings in Theorem 1.2 are over fields, over each completion $\mathbb{Z}_p$, and over finite coefficient rings.

7. Aomoto nonresonance over every field

Let $A_k^* = H^*(M;k)$, with meridian classes $e_H \in A_k^1$. For scalars $\lambda_H \in k$ set

$$\omega = \sum_H \lambda_H e_H, \qquad \lambda_\infty = -\sum_H \lambda_H, \qquad \lambda_X = \sum_{H \supseteq X} \lambda_H.$$

The *Aomoto complex* is $(A_k^*, \omega \wedge -)$. The Orlik–Solomon presentation ensures $\omega^2 = 0$ in every characteristic.

Theorem 7.1. *If $\lambda_X \neq 0$ for every dense edge $X \subseteq H_\infty$, then, for every field k,*

$$H^i(A_k^*, \omega \wedge -) = 0 \quad (i \neq \ell), \qquad \dim_k H^\ell(A_k^*, \omega \wedge -) = \beta_\ell.$$

Proof. Take $\mathcal{O} = k[[s]]$ and $q_H = 1 + s\lambda_H$. Then $q_X = 1 + s\lambda_X + O(s^2)$, so Theorem 2.1 applies. On a minimal CW model its completed cochain differential is $d = sD$. Since $d^2 = 0$ and $\mathcal{O}$ is a domain, $D^2 = 0$. The first-order identification of the universal abelian cellular complex [6, Theorem 12.6] identifies $(C^*/sC^*, D \bmod s)$ with the Aomoto complex. This identification holds over every field. An opposite left-right convention replaces ω by $-\omega$, which has the same cohomology.

Decompose (C^*, d) by Lemma 5.1. Theorem 2.1 and minimality leave only s blocks and β_ℓ isolated top terms. The same changes of basis turn $D = d/s$ into unit blocks and those isolated top terms. After reduction modulo s, the unit blocks remain contractible. The isolated terms give precisely k^{β_ℓ} at the top. □

Bailet–Yoshinaga [1] prove the corresponding unit-weight statement for real arrangements over arbitrary commutative rings; the arbitrary complex characteristic-zero case is also classical. Theorem 7.1 supplies the all-field assertion for arbitrary complex arrangements, including positive characteristic. The first-order cellular identification converts the valuation-one local-system theorem into this Aomoto vanishing statement.

Corollary 7.2. *For a one-parameter local system satisfying (2.2), the s-adic cochain spectral sequence of a minimal model degenerates at its second page.*

Proof. The Smith changes of basis preserve every power of s. Each s block contributes only its first differential, and each isolated term contributes none. The direct-sum decomposition therefore has no higher differential. □

8. Examples and the role of the hypothesis

For a complement of r points in $\mathbb{C}$, one has $b_0 = 1$, $b_1 = r$, $\beta_0 = 1$, and $\beta_1 = r - 1$. The infinity condition says that the sum of the meridian weights is a unit modulo N. The theorem gives

$$H^1(Y;k) \cong k \oplus k[C_N]^{r-1}, \qquad \operatorname{rank} H^1(Y;\mathbb{Z}) = 1 + N(r-1),$$

including every modular characteristic. This agrees with the Euler characteristic of the covering graph.

For a central arrangement with a prime number N of hyperplanes, the reduced Milnor fibre is the cyclic N-cover of a decone with all projective meridian weights equal to one. At an infinity edge its weight is its hyperplane multiplicity, strictly between zero and N whenever that edge is nonempty in projective space and the arrangement is essential. It therefore satisfies the condition, and its integral cohomology is free.

Valuation one is the step that excludes thick torsion. Already for $\mathbb{C}^*$ and a DVR local system with monodromy $1 + s^u$, the cochain complex is $\mathcal{O} \xrightarrow{s^u} \mathcal{O}$ and its first cohomology is $\mathcal{O}/(s^u)$. For $u > 1$ this module has one generator but length u. The valuation-one hypothesis makes these two quantities equal in the induction. For the cyclic deformation $q_H = (1+s)^{m_H}$, Xie's CDO condition supplies that hypothesis at every prime dividing N.

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