# CLIQUE RECONSTRUCTION AND INTEGRAL LIE STRUCTURES

## COHEN–FALK'S PROBLEM 5.13 AND THE EXTENSION TO CLOSED SURFACES

DENZEL ZHENG

Abstract. For every finite simple graph, its planar graphic arrangement group is reconstructed from its complete-subgraph pure braid groups. In particular, the product of the forgetful homomorphisms to maximal cliques is injective, answering both parts of Cohen–Falk's Problem 5.13. The same reconstruction holds for graph configuration groups on every closed orientable surface of positive genus. A finite elimination procedure proves these statements directly in discrete groups. We also reconstruct the integral lower-central Lie ring from the clique diagram and prove that its embedding into the product over maximal cliques is saturated. The surface argument uses a root-intersection formula in free Lie algebras over every field, including characteristic two, followed by integral base-change and saturation arguments. This yields torsion-free lower-central quotients in all degrees and an exact clique formula for their ranks. At fixed genus, the complete rank sequence determines, and is determined by, the clique polynomial.

## 1. The question and the reconstruction theorems

Let $\Gamma = (V, E)$ be a finite simple graph and let $M$ be a connected surface. Fix distinct basepoints $(p_v)_{v\in V}$ and put

$$X_\Gamma(M) = \{(z_v) \in M^V : z_i \neq z_j \text{ if } ij \in E\}, \qquad G_\Gamma(M) = \pi_1(X_\Gamma(M), (p_v)).$$

For $T \subseteq V$, write $P_T(M) = \pi_1 \operatorname{Conf}_T(M)$, with the retained basepoints, and set $P_\varnothing(M) = 1$. The graph is finite throughout. Let $\mathcal{C}(\Gamma)$ be its nonempty cliques, including singleton cliques, and let $\mathcal{M}(\Gamma)$ be its inclusion-maximal cliques. All maps in the clique diagram are forgetful maps, with labels and basepoints retained.

When $M = \mathbb{C}$, the group $G_\Gamma(\mathbb{C})$ is the graphic arrangement group associated with the hyperplanes $z_i = z_j$, $ij \in E$. Cohen–Falk [5, Problem 5.13] ask which graphs make the specified maximal-clique map injective and which graphic arrangement groups embed into products of pure braid groups. The first theorem gives the answer to both questions. Its proof also explains how compatible clique data determine an actual group element.

**Theorem 1.1** (Discrete clique reconstruction)**.** *Let $M = \mathbb{C}$ or let $M = \Sigma_g$ be a closed connected orientable surface of genus $g \geq 1$. For every finite simple graph $\Gamma$, the natural map*

$$\rho_\Gamma : G_\Gamma(M) \longrightarrow \prod_{C\in\mathcal{M}(\Gamma)} P_C(M)$$

*is injective. More precisely, there is a natural isomorphism*

$$G_\Gamma(M) \cong \varprojlim_{T\in\mathcal{C}(\Gamma)} P_T(M). \tag{1}$$

*Thus the image in the maximal-clique product consists exactly of the tuples whose restrictions agree on every clique overlap.*

*Date*: Research note / October 2026.

2020 *Mathematics Subject Classification.* 20F36, 20F14, 17B55, 55R80.

*Key words and phrases.* Graphic arrangement, surface configuration space, Brunnian braid, clique, lower central series, integral Lie ring, higher Hopf formula.

For a group $H$, put $\gamma_1 H = H$, $\gamma_{d+1}H = [H, \gamma_d H]$, and $\operatorname{gr}_{\mathbb{Z}} H = \bigoplus_{d\geq 1} \gamma_d H/\gamma_{d+1}H$, with its ordinary integral Lie bracket. A subgroup of a free abelian group is *saturated* if its quotient has no torsion.

**Theorem 1.2** (Integral clique reconstruction)**.** *In the range of Theorem 1.1, the forgetful maps induce a natural isomorphism of graded Lie rings*

$$\operatorname{gr}_{\mathbb{Z}} G_\Gamma(M) \cong \varprojlim_{T\in\mathcal{C}(\Gamma)} \operatorname{gr}_{\mathbb{Z}} P_T(M). \tag{2}$$

*Every homogeneous group on the left is free abelian of finite rank. Its image in the product over maximal cliques is saturated in every degree. Furthermore, for every $d \geq 1$, the discrete embedding is strict for the ordinary lower central series:*

$$\rho_\Gamma(G_\Gamma(M)) \cap \prod_{C\in\mathcal{M}(\Gamma)} \gamma_d P_C(M) = \rho_\Gamma(\gamma_d G_\Gamma(M)). \tag{3}$$

The planar proof preserves arbitrary verbal subgroups, whereas the surface proof of Theorem 1.2 takes place in graded Lie rings and requires an additional integral argument. In particular, the proof of the group theorem does not assume the integral theorem.

For the rank formula, let $c_s(\Gamma)$ be the number of $s$-vertex cliques. Define $a_{g,0,d} = 0$, and for $m \geq 1$ define $a_{g,m,d}$ by

$$\prod_{d\geq 1}(1-t^d)^{a_{g,m,d}} = U_{g,m}(t) := (1-2gt+t^2)\prod_{j=0}^{m-2}(1-2gt-jt^2). \tag{4}$$

An empty product is one; also set $U_{g,0} = 1$. Put

$$b_{g,s,d} = \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}a_{g,j,d}. \tag{5}$$

**Theorem 1.3** (Ranks and clique inversion)**.** *For $g \geq 1$, any finite simple graph $\Gamma$, and every $d \geq 1$, the rank of the integral lower-central quotient is*

$$\operatorname{rank}\operatorname{gr}_d G_\Gamma(\Sigma_g) = \sum_{s\geq 1} c_s(\Gamma) b_{g,s,d}. \tag{6}$$

*For $s \geq 2$,*

$$b_{g,s,d} = 0 \quad (d < 2s-2), \qquad b_{g,s,2s-2} = (s-2)!. \tag{7}$$

*Consequently, at a fixed genus the complete rank sequence and the clique polynomial determine one another.*

The complete-graph product in (4) is classical; the reconstruction and the identification of the nonedge quotient supply its extension to arbitrary graphs. The genus-one case permits triangles and uses the actual lower-central Lie ring, without an assumption of 1-formality. Nothing in these statements concerns arbitrary hyperplane arrangements, spherical configuration spaces, or restricted Lie algebras.

## 2. Deletion kernels and Brunnian lifting

For this section let $M = \mathbb{C}$ or $\Sigma_g$, $g \geq 1$, and let $P_V = P_V(M)$. Write $q_T : P_V \to P_T$ for retention of $T$. Set

$$J_i(V) = \ker(q_{V\setminus\{i\}}), \qquad J_T(V) = \bigcap_{i\in T} J_i(V), \qquad J_\varnothing(V) = P_V.$$

The Brunnian subgroup on a nonempty set $T$ is

$$\operatorname{Br}_T(M) = \bigcap_{i\in T} \ker(P_T \longrightarrow P_{T\setminus\{i\}}).$$

In particular $\mathrm{Br}_{\{i\}}(M) = \pi_1 M$. We use the pure deletion-kernel definition also for two points.

The Fadell–Neuwirth fibration [8] gives a surjective forgetful map $P_V \to P_{V\setminus\{v\}}$. Its kernel is $\pi_1(M \setminus \{p_i : i \neq v\})$: the fiber fundamental group injects because the base complete configuration space has $\pi_2 = 0$. For the surfaces considered here this vanishing follows inductively from the same fibration, starting with the asphericity of $M$ and its punctured surfaces. This observation is about complete configuration spaces and is not an asphericity assertion for $X_\Gamma(M)$.

Let $A_{ij} \in P_V$ be a based meridian of the diagonal $z_i = z_j$. Different paths to that diagonal change its conjugacy class representative, which does not change its normal closure.

**Lemma 2.1** (The two-deletion kernel)**.** *For distinct $i, j \in V$,*

$$J_i(V) \cap J_j(V) = \langle\langle A_{ij} \rangle\rangle_{P_V}. \tag{8}$$

*Proof.* Identify $J_j(V)$ with the fundamental group of $M$ punctured at the other basepoints. Deletion of $i$ on this fiber fills in the puncture $p_i$. The induced fundamental-group map has kernel normally generated by the peripheral loop about $p_i$, by van Kampen. That loop maps to $A_{ij}$ up to conjugacy and inversion. Hence $J_i \cap J_j = \langle\langle A_{ij} \rangle\rangle_{J_j}$. The left side is already normal in $P_V$. The inclusions of the normal closure in $J_j$, the normal closure in $P_V$, and the normal subgroup $J_i \cap J_j$ give equality. This also proves the two-point case. □

For normal subgroups $R_1, \dots, R_r$, the notation $[R_1, \dots, R_r]_{\mathrm{sym}}$ denotes the product over permutations of their left-nested iterated commutator subgroups. With one entry it denotes that entry. The following is the precise classical input needed for surfaces: if $M \neq S^2, \mathbb{RP}^2$ and $|T| \geq 3$, then, for any $v \in T$,

$$\mathrm{Br}_T(M) = [\langle\langle A_{iv} \rangle\rangle_{P_T} : i \in T \setminus \{v\}]_{\mathrm{sym}}. \tag{9}$$

This is Bardakov–Mikhailov–Vershinin–Wu [2, Theorem 1.1 and Proposition 3.6]; their normal closures are taken in the pure braid group. For two points we use Lemma 2.1, rather than the convention for two-strand Brunnian elements in a full braid group.

**Lemma 2.2** (Brunnian lifting)**.** *For every nonempty $T \subseteq V$,*

$$q_T(J_T(V)) = \mathrm{Br}_T(M). \tag{10}$$

*Proof.* The inclusion from left to right follows by commuting the forgetful maps. For a singleton it is the surjection induced by filling all punctures of $M$. Suppose $|T| \geq 2$ and choose $v \in T$. In the source group put $R_i = \langle\langle A_{iv} \rangle\rangle_{P_V}$ for $i \in T \setminus \{v\}$. Lemma 2.1 gives $R_i = J_i \cap J_v$. The subgroup $D = [R_i : i \in T \setminus \{v\}]_{\mathrm{sym}}$ therefore lies in $J_T(V)$: each commutator factor lies in every normal subgroup whose label occurs in it. A surjective homomorphism takes a normal closure onto the normal closure of its image, and it commutes with commutator subgroups. Thus $q_T(D)$ is the right side of (9), hence $\mathrm{Br}_T(M)$. For $|T| = 2$ the same argument has one root and uses Lemma 2.1 in the target. □

The planar version of this lemma admits a stronger direct geometric proof. It is important for preserving group filtrations.

**Lemma 2.3** (Planar disk sections)**.** *For $M = \mathbb{C}$ and every $T \subseteq V$ there is a group homomorphism $s_T : P_T(\mathbb{C}) \to P_V(\mathbb{C})$ such that*

$$q_T s_T = \mathrm{id}, \qquad s_T(\mathrm{Br}_T(\mathbb{C})) \subseteq J_T(V).$$

*Moreover, deletion of any $i \in T$ after $s_T$ factors through deletion of $i$ in $P_T$, with a homomorphism defined using the same disk. The sections for different $T$ need not be compatible with one another.*

*Proof.* Choose an embedded tree joining the selected basepoints and avoiding all unselected basepoints. A sufficiently small smooth regular neighborhood is a Jordan disk $U$ containing exactly the selected points. The inclusion $\mathrm{Conf}_T(U) \to \mathrm{Conf}_T(\mathbb{C})$ is a based homotopy equivalence. To see

this, use a Schoenflies homeomorphism to identify $U$ with the unit disk, take $r_0 < 1$ containing the selected points, and choose a strictly increasing continuous function $f : [0, \infty) \to [0, 1)$ with $f(r) = r$ for $r \le r_0$ and $\lim_{r\to\infty} f(r) = 1$. The radial maps with radius $(1 - u)r + uf(r)$, $0 \le u \le 1$, are embeddings, fix the basepoints, and keep the unit disk inside itself. Applied coordinatewise, they give a based homotopy inverse to the inclusion, also after any selected points are deleted.

Now keep all unselected strands fixed outside $U$. The resulting homomorphism $P_T(U) \to P_V(\mathbb{C})$, composed with the inverse of the inclusion isomorphism, is $s_T$. If $b \in \mathrm{Br}_T$, deletion of a selected strand is already null-homotopic inside $U$, since the smaller disk configuration inclusion is also a fundamental-group isomorphism. Keeping the outside points fixed extends this null-homotopy. This proves $s_T(b) \in J_T(V)$ and the asserted deletion factorization. □

## 3. Finite elimination and the actual graph group

Choose, for each nonempty $T$, a set-theoretic right inverse $\sigma_T : \mathrm{Br}_T \to J_T(V)$ to $q_T$, with $\sigma_T(1) = 1$. Lemma 2.2 supplies it. No homomorphism property is assumed. Order all nonempty subsets of $V$ by increasing cardinality, breaking ties arbitrarily.

**Lemma 3.1** (Finite multiplicative elimination)**.** *Every $h \in P_V$ has a finite factorization*

$$h = \prod_T c_T, \qquad c_T \in \sigma_T(\mathrm{Br}_T),$$

*in the chosen order. If $q_C(h) = 1$ for every clique $C$ of $\Gamma$, then every clique factor $c_T$ equals one.*

*Proof.* Start with residual element $r = h$. When $T$ is reached, every previously processed projection is one. In particular, deletion of each $i \in T$ from $q_T(r)$ gives one, so $q_T(r) \in \mathrm{Br}_T$. Set $c_T = \sigma_T(q_T(r))$ and replace $r$ by $c_T^{-1} r$. This kills the $T$ projection. A previous subset $S$ does not contain $T$, so choose $i \in T \setminus S$. Since $c_T \in J_i$, $q_S(c_T) = 1$, and all previous projections remain one. At $T = V$ the residual element becomes one. Tracking the left multiplications gives the displayed factorization in exactly the stated order.

Under the additional hypothesis, argue by the same order. Before a clique $T$ is processed, the nontrivial previous factors can only come from noncliques $U$, by induction. Such a $U$ cannot be contained in $T$, so $q_T$ kills $c_U \in J_U$. The current $T$ projection is therefore the original $q_T(h) = 1$, giving $c_T = \sigma_T(1) = 1$. □

Put

$$N_\Gamma = \langle\langle A_{ij} : ij \notin E,\ i \ne j \rangle\rangle_{P_V}.$$

For every nonclique $T$, Lemma 2.1 gives $J_T \subseteq N_\Gamma$, by choosing a nonedge inside $T$. Every clique projection kills $N_\Gamma$. Finite elimination consequently gives the exact discrete kernel identity

$$\bigcap_{C \in \mathcal{M}(\Gamma)} \ker(q_C : P_V \to P_C) = N_\Gamma. \tag{11}$$

**Lemma 3.2** (Allowing nonedge collisions)**.** *The inclusion $\mathrm{Conf}_V(M) \subseteq X_\Gamma(M)$ induces a surjection of fundamental groups with kernel $N_\Gamma$. Thus*

$$G_\Gamma(M) = P_V(M)/N_\Gamma. \tag{12}$$

*Proof.* A loop in $X_\Gamma(M)$ can be perturbed off all allowed pairwise diagonals, since these have real codimension two. This proves surjectivity. A null-homotopy of a loop in the complete configuration space can be made transverse to those diagonals and disjoint from their intersection strata, which have real codimension at least four. Removing small disks about the finitely many transverse intersections expresses the loop as a product of conjugates of diagonal meridians. Each top-dimensional diagonal stratum is connected: after merging its two coordinates, it is a connected product of surfaces with finitely many codimension-two diagonals removed. Hence its meridians are conjugate to the chosen

$A_{ij}$. Conversely, each such meridian becomes null-homotopic when the corresponding collision is allowed. Van Kampen gives precisely $N_\Gamma$ as the kernel. □

*Proof of Theorem 1.1.* The injectivity follows by combining (11) with (12). For the inverse-limit assertion, let $(a_T)_{T\in\mathcal{C}(\Gamma)}$ be a compatible family. Starting with $h = 1$ in $P_V$, process the cliques by increasing cardinality. At clique $T$, put $d_T = q_T(h)^{-1}a_T$. Compatibility and the previously matched proper subcliques give $d_T \in \mathrm{Br}_T$. Choose $c_T \in J_T$ with $q_T(c_T) = d_T$ and replace $h$ by $hc_T$. This matches $T$ and keeps every earlier clique unchanged, because none contains $T$. The process is finite and constructs a preimage of the family. Quotienting by its already identified kernel proves (1).

Finally, compatible maximal-clique tuples determine their values on every smaller clique. Agreement on each overlap is exactly the consistency needed for these restrictions. Thus the described image is exact. □

For $V = \varnothing$, all groups and limits are trivial. Singletons must be included for closed surfaces, since isolated vertices contribute copies of $\pi_1\Sigma_g$.

## 4. Planar verbal strictness and integral consequences

A verbal subgroup $W(H)$ is generated by the values in $H$ of a fixed family of group words. It is normal and is preserved by every group homomorphism. The same proof works for any subgroup functor with these properties.

**Proposition 4.1** (Planar verbal strictness)**.** *For $M = \mathbb{C}$ and every verbal subgroup functor $W$,*

$$\rho_\Gamma(G_\Gamma(\mathbb{C})) \cap \prod_{C\in\mathcal{M}(\Gamma)} W(P_C) = \rho_\Gamma(W(G_\Gamma(\mathbb{C}))). \tag{13}$$

*In particular $G_\Gamma(\mathbb{C})/W(G_\Gamma(\mathbb{C}))$ embeds into $\prod_C P_C/W(P_C)$.*

*Proof.* Take an element on the left and its full compatible clique family $(a_T)$. Every $a_T$ belongs to $W(P_T)$. Repeat the matching procedure in the proof of Theorem 1.1, now using the homomorphisms $s_T$ of Lemma 2.3. At each step, if $h \in W(P_V)$, then $d_T = q_T(h)^{-1}a_T \in \mathrm{Br}_T \cap W(P_T)$ and $s_T(d_T) \in J_T \cap W(P_V)$. Thus $h \leftarrow hs_T(d_T)$ stays in $W(P_V)$ and preserves all previous matches. At the end the image of $h$ in $G_\Gamma$ has the given clique family, hence is the original element by the proved group injection. Its membership in $W(G_\Gamma)$ follows by functoriality. The reverse inclusion is immediate. The quotient injection is exactly the kernel assertion in (13). □

**Proposition 4.2.** *Theorem 1.2 holds for $M = \mathbb{C}$.*

*Proof.* Taking $W = \gamma_d$ in Proposition 4.1 gives discrete strictness. Taking it also for $\gamma_{d+1}$ proves injectivity of the degree-$d$ graded comparison. The complete planar pure braid groups have free abelian lower-central quotients of finite rank, by the classical integral infinitesimal braid presentation, or their iterated almost-direct free-group decomposition [10, 9]. Hence the source graded groups are torsion-free; finite rank follows from finite generation.

To prove graded inverse-limit surjectivity, let $(x_T)$ be a compatible family in a fixed degree $d$. Match the cliques additively, starting from zero in $\mathrm{gr}_d P_V$. The current discrepancy at $T$ lies in the intersection of the single-deletion kernels in $\mathrm{gr}_d P_T$. Lemma 2.3 supplies a lift under $\mathrm{gr}_d(s_T)$ lying in all the corresponding source deletion kernels, because its deletion factorization is an actual group factorization. Adding this lift preserves earlier matches. The finite procedure gives a preimage. This argument never identifies gr of a group-kernel intersection with an intersection of graded kernels without proof.

The image in the maximal-clique product is therefore the kernel of the homomorphism recording differences of restrictions to overlaps. Its target is a finite product of free abelian degree-$d$ groups. The quotient by that kernel is an image subgroup of a free abelian group, hence is torsion-free. This proves saturation. □

The discrete theorem has several direct consequences, which use only classical properties of the clique factors.

**Corollary 4.3.** *Every planar graphic arrangement group is linear over a characteristic-zero field and is residually torsion-free nilpotent. Every graph configuration group on $\Sigma_g$, $g \geq 1$, is residually torsion-free nilpotent. All these groups are torsion-free and residually finite.*

*Proof.* Restrict the Lawrence–Krammer faithful representation of each full planar braid group [11] to its pure subgroup and take a block-diagonal sum over maximal cliques. For a graph with an edge this gives dimension $\sum_C \binom{|C|}{2}$ over $\mathbb{Q}(q,t)$. The edgeless planar group is trivial. Planar pure braid groups are residually torsion-free nilpotent [9]; closed positive-genus pure braid groups have this property by [1, Corollary 7]. Finite products and subgroups preserve it. For residual finiteness, a nontrivial element can first be detected in a torsion-free nilpotent quotient. The image of the finitely generated source there is a finitely generated nilpotent group, and such groups are residually finite. Torsion-freeness follows from the separating torsion-free quotients. □

The free abelian lower-central quotients were proved separately in Proposition 4.2. Residual torsion-free nilpotence alone is not used to infer that assertion.

## 5. A Lie root-intersection formula over every field

The remaining integral proof for surfaces requires information in every characteristic. Throughout this section, a Lie algebra over a field $k$ means a bilinear algebra satisfying Jacobi and $[x,x]=0$ for every $x$. In characteristic two this alternating identity is part of the definition; we do not use a restricted $p$-operation.

For Lie ideals $R_1,\dots,R_r$, let $D(R_1,\dots,R_r)$ be the sum of all iterated bracket ideals in which every listed ideal occurs. It equals the sum over permutations of left-nested brackets, denoted $[R_1,\dots,R_r]_{\mathrm{sym}}$. Repeated entries in the bracket expression do not enlarge this ideal.

**Lemma 5.1** (Coordinate ideals)**.** *In $F = \mathrm{FreeLie}_k(W,t_1,\dots,t_m)$, put $T_i = \langle t_i \rangle_{\mathrm{Lie},F}$. For $A \subseteq \{1,\dots,m\}$, the intersection $\bigcap_{i\in A} T_i$ is spanned by Lie monomials containing every indicated coordinate and is equal to $D(T_i : i \in A)$. The quotient cube formed by any subfamily of these ideals has surjective boundary maps to its punctured-cube limits.*

*Proof.* The free Lie algebra has the nonnegative multigrading given by the number of occurrences of each $t_i$. The ideal $T_i$ is the sum of components whose $i$th entry is positive, proving the spanning assertion.

To identify the bracket ideal, mark one occurrence of each required label in a bracket tree. An unmarked branch can be moved to a marked branch using $[u,[v,w]] = [[u,v],w] + [v,[u,w]]$; it is thereby absorbed into the appropriate ideal entry. Induction on the tree length reduces the expression to brackets whose entries have the marked labels. Repeated applications of the same identity express any such bracket tree as a sum of left-nested brackets over permutations. All coefficients are integers with sign, so this works over every field. Conversely every such bracket belongs to each $T_i$ by ideality. The same Jacobi argument for arbitrary ideal entries gives the fat-bracket and symmetric-bracket equality used above.

For the boundary assertion, split by coordinate support. The quotient $F/T_i$ keeps exactly the components not containing $i$, and the quotient by $T_i + T_j$ keeps those containing neither. A compatible family therefore specifies the same value for a component wherever that component survives. Choose these common values in $F$, and choose zero for components seen in no quotient. Their finite sum lifts the family. The same reasoning after some coordinates have been removed proves every boundary assertion. □

We make explicit the homological input, including its characteristic-two interpretation. An $r$-fold projective presentation is an $r$-cube of surjections such that every nonterminal vertex is projective and every vertex maps surjectively to the limit of its punctured outgoing cube.

**Proposition 5.2** (Alternating binary Lie Hopf formula)**.** *Let $k$ be any field. Suppose the quotient cube of a free Lie algebra $F$ by ideals $R_1, \dots, R_r$ is an $r$-fold projective presentation of $L$. With an empty intersection interpreted as $F$, there is an isomorphism*

$$H_{r+1}^{\mathrm{CE}}(L;k) \cong \frac{[F,F] \cap \bigcap_{i=1}^{r} R_i}{\sum_{A \subseteq \{1,\dots,r\}} [\bigcap_{i \in A} R_i, \bigcap_{i \notin A} R_i]}. \tag{14}$$

*Proof.* We use the general higher Hopf theorem of Everaert–Gran–Van der Linden [7, Theorem 8.1, Remark 8.2 and Section 9.4]. Its hypotheses are a semi-abelian monadic category, a Birkhoff reflector, and a projective cube presentation. Alternating binary Lie algebras over any field form a variety of $\Omega$-groups, hence such a category; abelianization $L \mapsto L/[L,L]$ is the Birkhoff reflector. Free Lie algebras on vector spaces are projective because vector-space maps lift along surjections. The general theorem first gives the corresponding comonadic homology quotient. Two verifications identify it with (14).

First, its centralization denominator has precisely the displayed binary bracket form. For a surjection $B \to A$ with kernel $K$, the kernel pair is, as a vector space, $\Delta B + (0,K)$. On brackets, the difference of its two projections is generated by $[B,K]$; centralization therefore quotients $B$ by that ideal. Inductively take the kernel pair in the last direction of an $r$-cube. If $K_A$ denotes an intersection of the earlier direction kernels, its corresponding kernel-pair intersection is $\Delta K_A + (0, K_A \cap R_r)$. For a partition of the earlier directions into $A$ and its complement, expanding the bracket of two such spaces gives projection differences generated by

$$[K_A, K_{A^c} \cap R_r] + [K_A \cap R_r, K_{A^c}].$$

The term with both entries in $R_r$ is included in these ideals. Summing over the partitions, and starting with the one-direction calculation, gives exactly the denominator of (14) as the higher centralization kernel in the general theorem. Only bilinearity was used; no division by two or polarization of the alternating identity occurs.

Second, comonadic homology here agrees, with the usual one-degree shift, with CE homology. Use the free-Lie/underlying-vector-space comonad and its standard free simplicial resolution $P_\bullet \to L$. The augmented underlying simplicial vector space has a *linear* extra degeneracy and is split-contractible to $L$. Consider the first-quadrant double complex formed from the normalized simplicial direction and $\Lambda^q P_p$, $q \geq 1$, with the CE differential in the other direction. For fixed $q$, the exterior powers of the extra degeneracy satisfy the same simplicial contraction identities, so the horizontal homology is $\Lambda^q L$. This spectral sequence computes $H_*^{\mathrm{CE}}(L;k)$ in positive degrees.

If $P = \mathrm{FreeLie}_k(W)$, then $U(P) = T_k(W)$, and

$$0 \longrightarrow T_k(W) \otimes W \longrightarrow T_k(W) \longrightarrow k \longrightarrow 0$$

is a free resolution of the augmentation module. The standard PBW–CE resolution over a field gives $H_q^{\mathrm{CE}}(P;k) = 0$ for $q \geq 2$ and $H_1^{\mathrm{CE}}(P;k) = P_{\mathrm{ab}}$. Thus the other spectral sequence retains only $q = 1$ and gives $H_{r+1}^{\mathrm{CE}}(L;k) = H_r((P_\bullet)_{\mathrm{ab}})$. This is the comonadic homology in the general Hopf theorem. Both spectral sequences are first-quadrant; the truncation at $q = 1$ does not change CE homology in positive degrees, since the differential into $\Lambda^0 L$ is zero. PBW and its CE free resolution hold for alternating Lie algebras over every field [4, 13]. This completes both identifications, including characteristic two. □

**Lemma 5.3** (Surface Lie homology)**.** *For $g \geq 1$ put*

$$S_g(k) = \mathrm{FreeLie}_k(a_1, b_1, \ldots, a_g, b_g)/(c), \qquad c = \sum_{\ell=1}^{g} [a_\ell, b_\ell].$$

*Then $H_q^{\mathrm{CE}}(S_g(k); k) = 0$ for $q \geq 3$, over every field $k$, and*

$$\mathrm{Hilb}\, U(S_g(k)) = \frac{1}{1 - 2gt + t^2}. \tag{15}$$

*Proof.* The enveloping algebra is the tensor algebra on the $2g$ generators modulo $\sum_\ell (a_\ell b_\ell - b_\ell a_\ell)$. Choose a degree-lexicographic order whose leading word is $a_1 b_1$. This word has no proper overlap with itself, because $a_1 \neq b_1$, and the leading coefficient is one. Reduction by this single relation therefore has no overlap ambiguities. The normal words avoid $a_1 b_1$; their number satisfies the recurrence giving (15). This elementary instance of the diamond lemma [3] also applies in characteristic two.

Let $U = U(S_g(k))$. There is a homogeneous complex of left $U$-modules

$$0 \longrightarrow U(-2) \xrightarrow{d_2} U(-1)^{2g} \xrightarrow{d_1} U \longrightarrow k \longrightarrow 0, \tag{16}$$

where

$$d_2(u) = (-ub_\ell, ua_\ell)_{\ell=1}^{g}, \qquad d_1((v_\ell, w_\ell)_\ell) = \sum_\ell (v_\ell a_\ell + w_\ell b_\ell).$$

The relation gives $d_1 d_2 = 0$, and the image of $d_1$ is the augmentation ideal, by splitting every nonempty word at its last letter. The ordinary PBW filtration has associated graded algebra $\mathrm{Sym}_k(S_g(k))$, a domain. Therefore $U$ is a domain. As $a_1 \neq 0$, the equality $d_2(u) = 0$ implies $ua_1 = 0$ and $u = 0$. Finally,

$$\mathrm{Hilb}\ker d_1 = 2gt\,\mathrm{Hilb}\, U - (\mathrm{Hilb}\, U - 1) = t^2\,\mathrm{Hilb}\, U,$$

which equals the Hilbert series of $\mathrm{im}\, d_2$. The inclusion of image in kernel and finite-dimensional homogeneous pieces prove exactness at the middle term. Thus (16) is a length-two free resolution. Its higher Tor groups vanish, proving the CE assertion. □

**Theorem 5.4** (Surface roots)**.** *Let $g, r \geq 1$ and let $k$ be any field. In*

$$F_r = \mathrm{FreeLie}_k(a_1, b_1, \ldots, a_g, b_g, t_1, \ldots, t_{r-1}),$$

*give the $a_\ell, b_\ell$ weight one and the $t_i$ weight two. Put*

$$R_0 = \langle c + \sum_{i=1}^{r-1} t_i \rangle_{\mathrm{Lie}}, \qquad R_i = \langle t_i \rangle_{\mathrm{Lie}} \quad (1 \leq i \leq r-1).$$

*Then*

$$\bigcap_{i=0}^{r-1} R_i = [R_0, \ldots, R_{r-1}]_{\mathrm{sym}}. \tag{17}$$

*Proof.* Any proper subfamily of the roots can be made into coordinate ideals. If it contains $R_0$, choose a missing $j \geq 1$ and replace the free coordinate $t_j$ by $t_j' = c + \sum_i t_i$. This is an invertible free change of variables, with inverse $t_j = t_j' - c - \sum_{i \neq j} t_i$. Lemma 5.1 consequently identifies the intersection for every proper subfamily with its symmetric bracket ideal.

We prove (17) and the exact-cube condition simultaneously by induction on $r$. For $r = 1$ the intersection assertion is tautological and the cube is the surjection $\mathrm{FreeLie}_k(a_\ell, b_\ell) \to S_g(k)$. Suppose the assertions hold for smaller root families. Form the cube

$$f_A = F_r / \sum_{i \in A} R_i \qquad (A \subseteq \{0, \ldots, r-1\}).$$

All nonterminal vertices are free: if $0 \notin A$ one deletes coordinates, and if $0 \in A$ one eliminates an undeleted coordinate using the surface relation.

To verify the initial boundary surjection, take compatible elements in the single-root quotients. First lift their values in the coordinate quotients, $i \geq 1$, to $z \in F_r$ by Lemma 5.1. Their prescribed $R_0$ value differs from the image of $z$ by

$$\delta \in \bigcap_{i \geq 1} \operatorname{im}(R_i \to F_r/R_0).$$

In $F_r/R_0$ eliminate $t_{r-1}$. The displayed image ideals become the smaller surface root family

$$(t_1), \ldots, (t_{r-2}), (-c - \sum_{i=1}^{r-2} t_i).$$

By induction its intersection is its symmetric bracket ideal. Each of these bracket expressions lifts from the corresponding $R_i$, so $\delta$ lifts to $w \in \bigcap_{i \geq 1} R_i$. Replacing $z$ by $z + w$ corrects the $R_0$ value and leaves all coordinate values unchanged. Pairwise compatibility suffices here: values at higher vertices of the quotient cube are determined by their images from the single-root vertices.

The outgoing cube of any other nonterminal vertex is a smaller surface root cube after deleted coordinates have been removed and, when necessary, one remaining coordinate has been eliminated. Its boundary maps are surjective by the same induction. Thus $f$ is genuinely an $r$-fold projective presentation, with terminal object $S_g(k)$.

Write $N = \bigcap_i R_i$ and $D = [R_0, \ldots, R_{r-1}]_{\mathrm{sym}}$. For $r \geq 2$, $N$ has no weight-one or weight-two part. For $r = 2$ the two weight-two root lines are spanned by $t_1$ and $t_1 + c$ and are independent. For $r \geq 3$ two distinct coordinate root lines already have zero intersection. The abelianization of $F_r$ is concentrated in weights one and two, so $N \subseteq [F_r, F_r]$. By Lemma 5.3, $H^{\mathrm{CE}}_{r+1}(S_g(k); k) = 0$. Apply Proposition 5.2. For every proper nonempty partition, the intersections in its denominator are the corresponding symmetric bracket ideals, as shown above. Their bracket sum is $D$: bracket trees can be split at the last bracket, and the Jacobi argument in Lemma 5.1 supplies the reverse description. The empty and full partitions give $[F_r, N]$. Therefore $N = [F_r, N] + D$.

The quotient $N/D$ is positively weight-graded. If it were nonzero, choose its least nonzero weight. The equality $N/D = [F_r, N/D]$ would express that weight using a strictly smaller weight of $N/D$, since every weight in $F_r$ is positive. This is impossible. Hence $N = D$, completing the induction. □

## 6. Complete surface Lie rings and vertex-kernel distributivity

We now record precisely the complete-configuration input. For $g \geq 1$, Enriquez–Vershinin [6, Theorem 2.1] prove over $\mathbb{Z}$ that the ordinary lower central and weight filtrations on $P_V(\Sigma_g)$ coincide, and that the Lie ring has degree-one generators $a_{\ell,i}, b_{\ell,i}$, $1 \leq \ell \leq g$, $i \in V$, with the following relations. For $i \neq j$, introduce degree-two symbols $t_{ij} = t_{ji}$ through the first displayed relation:

$$[a_{\ell,i}, b_{m,j}] = \delta_{\ell m} t_{ij}, \tag{18}$$

$$[a_{\ell,i}, a_{m,j}] = [b_{\ell,i}, b_{m,j}] = 0, \tag{19}$$

$$\sum_{\ell=1}^{g} [a_{\ell,i}, b_{\ell,i}] = -\sum_{j \neq i} t_{ij}, \tag{20}$$

$$[a_{\ell,k}, t_{ij}] = [b_{\ell,k}, t_{ij}] = 0 \quad (k \notin \{i, j\}). \tag{21}$$

They also recall the integral weight-graded exact sequence for a forgetful fibration [6, pp. 3–4], established in [12]. Its kernel is the weight Lie ring of a punctured surface, free on $2g$ weight-one handle generators and $|V| - 2$ weight-two peripheral generators. Concretely, for $|V| \geq 2$, deletion of $v$ has kernel

$$\mathrm{FreeLie}_{\mathbb{Z}}(a_{1,v}, b_{1,v}, \ldots, a_{g,v}, b_{g,v}, t_{iv} : i \neq v, p), \tag{22}$$

where $p \neq v$ is any chosen omitted peripheral label and

$$t_{pv} = -\sum_{\ell}[a_{\ell,v}, b_{\ell,v}] - \sum_{i\neq v,p} t_{iv}. \tag{23}$$

Forgetting another point fills its puncture, so its action on this kernel kills that peripheral root and gives the analogous target kernel. These assertions use the weight filtration on the fiber, not its ordinary lower central series. The weighted free Lie identification can also be read from the usual weighted Magnus–Witt construction: give handle letters weight one and the retained peripheral letters weight two.

**Lemma 6.1** (Integral complete factors)**.** *Every degree of $L_V^{\mathbb{Z}} := \mathrm{gr}_{\mathbb{Z}} P_V(\Sigma_g)$ is free abelian of finite rank. The deletion exact sequences with kernel (22) remain exact after tensoring with any field. Their graded dimensions are independent of that field.*

*Proof.* For one point the presentation is $S_g(\mathbb{Z})$. Base change of this presentation gives $S_g(k)$. PBW and (15) imply

$$\operatorname{Hilb} U(S_g(k)) = \prod_{d\geq 1}(1-t^d)^{-\dim_k S_g(k)_d}.$$

The coefficients determine the exponents recursively, so those dimensions are the same over $\mathbb{Q}$ and every $\mathbb{F}_p$. In each degree $S_g(\mathbb{Z})$ is a finitely generated abelian group. Its dimension modulo $p$ equals its rational rank plus the number of $p$-primary invariant factors; equality for every prime excludes torsion. The degrees are therefore free abelian.

Induct on $|V|$ using the classical integral deletion exact sequence. The weighted free Lie ring in its kernel is free abelian in every degree by the integral Hall basis [13]. Its quotient is free in every degree by induction. The sequence splits as a sequence of abelian groups degree by degree, proving the middle term free and preserving exactness under every field base change. The dimensions of the weighted free kernel are determined by its free generators and their weights and are independent of the field. The induction gives the final assertion. □

Fix a field $k$ and write $L_V = L_V^{\mathbb{Z}} \otimes k$. Let $\mathfrak{J}_i(V)$ be the deletion kernel in $L_V$, and set $\mathfrak{J}_T(V) = \bigcap_{i\in T}\mathfrak{J}_i(V)$, $\mathfrak{J}_\varnothing(V) = L_V$. We use fraktur notation to distinguish these Lie ideals from the discrete group kernels.

**Lemma 6.2** (Vertex-support representation)**.** *The ideal $\mathfrak{J}_T(V)$ is spanned by values of Lie monomials in the degree-one vertex generators that contain every vertex of $T$. Consequently, if $T \subseteq S \subseteq V$, retention of $S$ satisfies*

$$q_S(\mathfrak{J}_T(V)) = \mathfrak{J}_T(S). \tag{24}$$

*If $T \not\subseteq S$, that image is zero.*

*Proof.* A monomial containing a vertex $i$ is killed by deleting $i$; thus every monomial on the stated list belongs to the intersection. For the other inclusion, singletons are immediate from the presentation: the deletion kernel is the ideal generated by that vertex's handle generators. Suppose $|T| \geq 2$ and choose $v \in T$. Work in the free kernel (22), justified over $k$ by Lemma 6.1.

If $T \neq V$, choose $p \notin T$ as the omitted peripheral root. For each $i \in T \setminus \{v\}$, the intersection with the deletion kernel of $i$ is exactly the coordinate ideal $(t_{iv})$ in this free kernel. Lemma 5.1 gives words containing all of those roots. Expanding $t_{iv} = [a_{1,i}, b_{1,v}]$ and all remaining roots in the vertex generators gives words containing every vertex of $T$.

If $T = V$, the full peripheral root family consists of the coordinate roots and (23). Theorem 5.4, with $r = |V| - 1$, identifies their intersection with their symmetric bracket ideal. Each factor is an ideal generated by a root labelled $i, v$. The bracket expressions therefore expand into monomials containing every vertex of $V$. This includes $|V| = 2$, where $r = 1$ and the single root is the surface relation in the free kernel.

For (24), represent a target element by the monomials just obtained and evaluate exactly those monomials in the source. Each such value belongs to $\mathfrak{J}_T(V)$ and has the desired projection. This is a lifting of expressions; no inclusion of a smaller surface Lie algebra as a Lie subalgebra is assumed. The vanishing when $T$ meets the forgotten set follows from commuting deletion maps. □

**Lemma 6.3** (Distributivity with vertex-kernel sums)**.** *For all $T, I \subseteq V$,*

$$\mathfrak{J}_T \cap \sum_{i\in I} \mathfrak{J}_i = \sum_{i\in I} \mathfrak{J}_{T\cup\{i\}}. \tag{25}$$

*Proof.* The right side is contained in the left. The empty-$I$ case is trivial, and if $I \cap T \neq \varnothing$ both sides equal $\mathfrak{J}_T$. Assume $I \cap T = \varnothing$ and induct on $|V|$. Choose $i_0 \in I$ and delete $i_0$. An element $z$ of the left side projects into

$$\mathfrak{J}_T(V \setminus \{i_0\}) \cap \sum_{i\in I\setminus\{i_0\}} \mathfrak{J}_i(V \setminus \{i_0\}).$$

By induction write its image as a sum of elements of $\mathfrak{J}_{T\cup\{i\}}(V \setminus \{i_0\})$. Lift each term to $\mathfrak{J}_{T\cup\{i\}}(V)$ by Lemma 6.2. The difference between $z$ and the sum of the lifts belongs to $\mathfrak{J}_T \cap \mathfrak{J}_{i_0}$, which is $\mathfrak{J}_{T\cup\{i_0\}}$. This proves the required decomposition. □

**Lemma 6.4** (Two-vertex Lie root)**.** *For $i \neq j$,*

$$\mathfrak{J}_i \cap \mathfrak{J}_j = \langle t_{ij} \rangle_{\mathrm{Lie}, L_V}. \tag{26}$$

*Proof.* If $|V| \geq 3$, work in $\mathfrak{J}_j$ and omit a peripheral root with label outside $\{i, j\}$. Deleting $i$ kills precisely the free coordinate $t_{ij}$, so its kernel is that coordinate ideal within $\mathfrak{J}_j$. This ideal is contained in the $L_V$-ideal generated by $t_{ij}$, which in turn is contained in $\mathfrak{J}_i \cap \mathfrak{J}_j$. The inclusions give equality. For $|V| = 2$, $\mathfrak{J}_j$ is free on its $2g$ handle generators, and deleting $i$ imposes the single surface relation $c_j = -t_{ij}$. The same inclusion argument applies. □

## 7. From fieldwise support to a saturated integral reconstruction

For each nonempty $T \subseteq V$, choose homogeneous vector-space complements

$$\mathfrak{J}_T = C_T \oplus \sum_{j\notin T} \mathfrak{J}_{T\cup\{j\}}. \tag{27}$$

These choices are made over a field, degree by degree. They are not asserted to be integral splittings or Lie subalgebras.

**Proposition 7.1** (Simultaneous support decomposition)**.** *Retention of $T$ identifies $C_T$ with $\mathfrak{B}_T := \bigcap_{i\in T} \ker(L_T \to L_{T\setminus\{i\}})$, and*

$$L_V = \bigoplus_{\varnothing\neq T\subseteq V} C_T, \qquad \mathfrak{J}_T = \bigoplus_{U\supseteq T} C_U. \tag{28}$$

*Proof.* Lemma 6.2 gives $q_T(\mathfrak{J}_T) = \mathfrak{B}_T$. The kernel of $q_T$ in $L_V$ is $\sum_{j\notin T} \mathfrak{J}_j$, by the presentation. Lemma 6.3 identifies its intersection with $\mathfrak{J}_T$ as the second summand of (27). Hence $q_T : C_T \to \mathfrak{B}_T$ is an isomorphism. Descending substitution in (27) shows that the indicated complements span each $\mathfrak{J}_T$. They span $L_V$ because the vertex-generated ideals sum to $L_V$.

Suppose $\sum_U c_U = 0$ is a relation between complements. Choose a nonzero term $c_T$ with the least cardinality. Apply $q_T$. Any nonzero term indexed by $U \not\subseteq T$ is killed, since it lies in a kernel belonging to a forgotten vertex. Terms with $U \subsetneq T$ are zero by minimality. Thus only $q_T(c_T) = 0$ remains, contrary to its injectivity on $C_T$. This proves directness and both equalities. □

Define $L_\Gamma^{\mathbb{Z}}$ by the complete surface Lie presentation (18)–(21) with the additional relations $t_{ij} = 0$ for nonedges. Let $L_\Gamma(k)$ be its base change to $k$. In the complete field algebra it is the quotient by the ideal

$$K_\Gamma = \sum_{ij \notin E} \langle t_{ij} \rangle_{\mathrm{Lie}}.$$

Lemmas 6.4 and Proposition 7.1 give

$$K_\Gamma = \bigoplus_{T \text{ not a clique}} C_T, \qquad L_\Gamma(k) \cong \bigoplus_{T \in \mathcal{C}(\Gamma)} C_T \tag{29}$$

as graded vector spaces.

**Proposition 7.2** (Fieldwise clique comparison)**.** *For every field $k$, the natural map*

$$j_k : L_\Gamma(k) \longrightarrow \prod_{C \in \mathcal{M}(\Gamma)} L_C(k)$$

*is injective and its image is the full clique inverse limit. Its degree-$d$ dimension is*

$$\dim_k L_\Gamma(k)_d = \sum_{s \geq 1} c_s(\Gamma) \dim_k \mathfrak{B}_{[s],d}. \tag{30}$$

*These dimensions are independent of $k$.*

*Proof.* In $L_V$, the common kernel of the clique projections is exactly the nonclique sum in (29). Indeed it contains that sum; for the converse, choose a nonzero clique component of least cardinality in an element supported on clique blocks and project to that clique, just as in the directness proof. This proves injectivity after quotienting by $K_\Gamma$.

To lift a compatible family in the clique diagram, match cliques by increasing cardinality using addition. The discrepancy at $T$ lies in $\mathfrak{B}_T$. Lemma 6.2 lifts it to $\mathfrak{J}_T(V)$, so the correction changes no previously matched clique. The resulting finite sum maps to the family, proving the image assertion.

The dimension formula follows from (29), since $C_T \cong \mathfrak{B}_T$ and the complete algebra is invariant under vertex relabelling. For the complete graph on $m$ vertices, the same formula says $\dim_k L_{[m],d} = \sum_{s=1}^{m} \binom{m}{s} \dim_k \mathfrak{B}_{[s],d}$. Lemma 6.1 makes its left side independent of the field; binomial inversion makes every Brunnian dimension, hence every dimension in (30), independent of the field. □

**Proposition 7.3** (Integral saturation)**.** *Every degree of $L_\Gamma^{\mathbb{Z}}$ is free abelian of finite rank and the natural map*

$$j_{\mathbb{Z}} : L_\Gamma^{\mathbb{Z}} \longrightarrow P := \prod_{C \in \mathcal{M}(\Gamma)} L_C^{\mathbb{Z}} \tag{31}$$

*is injective with saturated image. Its image is the integral clique inverse limit.*

*Proof.* The integral presentation commutes with field base change. In each degree it defines a finitely generated abelian group. Proposition 7.2 gives equality of its dimensions over $\mathbb{Q}$ and every $\mathbb{F}_p$. The invariant-factor argument in Lemma 6.1 excludes every torsion prime, proving it free abelian. Rational injectivity of $j_{\mathbb{Z}}$, combined with this torsion-freeness, proves integral injectivity.

Let $D = P/j_{\mathbb{Z}}(L_\Gamma^{\mathbb{Z}})$ in a fixed degree. The product $P$ is free abelian, and $j_{\mathbb{Z}} \otimes \mathbb{F}_p = j_{\mathbb{F}_p}$ is injective for every prime by Proposition 7.2. The Tor sequence therefore gives

$$D[p] = \mathrm{Tor}_1^{\mathbb{Z}}(D, \mathbb{F}_p) = 0 \quad \text{for every prime } p.$$

This proves saturation.

Let $K \subseteq P$ be the kernel of all overlap restriction differences, that is, the integral clique inverse limit in the maximal-clique product. It contains the image of $j_{\mathbb{Z}}$, and rationalization, which is exact and commutes with this finite kernel, identifies $K \otimes \mathbb{Q}$ with that rational image by Proposition 7.2. Thus $K/j_{\mathbb{Z}}(L_\Gamma^{\mathbb{Z}})$ is a torsion group. It is also a subgroup of the torsion-free group $D$, and hence

vanishes. This proves the precise integral image. The proof does not interchange finite inverse limits with reduction modulo $p$ without justification. □

*Proof of Theorem 1.2 for closed surfaces.* By Lemma 3.2, the actual group $G_\Gamma$ is the quotient of $P_V$ by the normal closures of nonedge meridians. In the classical integral complete presentation, such a meridian lies in $\gamma_2 P_V$ and has initial class $\pm t_{ij}$. Thus the quotient induces a surjective graded Lie homomorphism

$$L_\Gamma^{\mathbb{Z}} \twoheadrightarrow \operatorname{gr}_{\mathbb{Z}} G_\Gamma(\Sigma_g).$$

The actual clique projections induce the other homomorphism in

$$L_\Gamma^{\mathbb{Z}} \twoheadrightarrow \operatorname{gr}_{\mathbb{Z}} G_\Gamma(\Sigma_g) \longrightarrow \prod_{C\in\mathcal{M}(\Gamma)} \operatorname{gr}_{\mathbb{Z}} P_C(\Sigma_g).$$

Their composite is $j_{\mathbb{Z}}$, injective by Proposition 7.3. The surjection is consequently an isomorphism, and the second arrow is injective with the same saturated inverse-limit image. This proves the Lie-ring and torsion assertions.

For strictness, suppose every clique image of $h \in G_\Gamma$ belongs to $\gamma_d P_C$. Inductively show $h \in \gamma_e G_\Gamma$ for $1 \le e \le d$. If $e < d$ and this holds for $e$, its degree-$e$ class maps to zero in every clique Lie ring. The just-proved graded injection makes that class zero, so $h \in \gamma_{e+1} G_\Gamma$. The finite induction proves (3); the opposite inclusion is functorial. No section of a closed-surface group and no infinite approximation is needed. □

## 8. The rank formula and recovery of the clique polynomial

Write $a_{g,m,d} = \operatorname{rank} \operatorname{gr}_d P_{[m]}(\Sigma_g)$. The complete deletion exact sequence contributes a free weighted Lie kernel on $2g$ degree-one and $m-2$ degree-two generators when one passes from $m-1$ to $m$ points. Its enveloping algebra is the tensor algebra on those generators, with Hilbert series $(1-2gt-(m-2)t^2)^{-1}$. PBW, or a basis of the kernel followed by a basis of the quotient in PBW order, multiplies the complete enveloping Hilbert series by this factor. Starting with Lemma 5.3 gives exactly (4).

*Proof of (6) in Theorem 1.3.* For a complete graph on $m$ vertices, Proposition 7.1 gives

$$a_{g,m,d} = \sum_{s=1}^{m} \binom{m}{s} \dim_{\mathbb{Q}} \mathfrak{B}_{[s],d}.$$

Binomial inversion identifies $\dim_{\mathbb{Q}} \mathfrak{B}_{[s],d}$ with $b_{g,s,d}$ in (5). The arbitrary-graph decomposition in Proposition 7.2 then gives the formula over $\mathbb{Q}$. Theorem 1.2 identifies those dimensions with the ranks of the actual free abelian lower-central quotients, proving the integral rank assertion. □

Equivalently, the full Euler product for the graph group is

$$\prod_{d\ge1}(1-t^d)^{\operatorname{rank}\operatorname{gr}_d G_\Gamma(\Sigma_g)} = \prod_{s\ge1}\left(\prod_{j=0}^{s} U_{g,j}(t)^{(-1)^{s-j}\binom{s}{j}}\right)^{c_s(\Gamma)}. \tag{32}$$

There are only finitely many nonzero clique counts.

*Proof of (7) and clique recovery.* It is useful to write, including $m=0$,

$$U_{g,m}(t) = \prod_{j=-1}^{m-2}(1-2gt-jt^2).$$

Let $P_{m,d} = [t^d](-\log U_{g,m}(t))$ and $p_d(j) = [t^d](-\log(1-2gt-jt^2))$. The latter is a polynomial in $j$ of degree at most $\lfloor d/2 \rfloor$. Hence

$$P_{m,d} = \sum_{j=-1}^{m-2} p_d(j)$$

is a polynomial in $m$ of degree at most $\lfloor d/2 \rfloor + 1$, with value zero at $m = 0$. This follows, for example, by expressing $p_d$ in the binomial-polynomial basis and taking its discrete antiderivative. Taking logarithms in the complete Euler product and applying divisor Möbius inversion gives

$$P_{m,d} = \frac{1}{d} \sum_{e|d} e\, a_{g,m,e}, \qquad a_{g,m,d} = \sum_{e|d} \frac{\mu(e)}{e} P_{m,d/e}. \tag{33}$$

By (5), $b_{g,s,d}$ is the $s$th forward difference of $a_{g,m,d}$ at $m = 0$. If $d < 2s-2$, each polynomial $P_{m,d/e}$ on the right of (33) has degree less than $s$. Its difference vanishes, proving the first assertion.

At $d = 2s-2$, all divisor terms with $e > 1$ still vanish under the $s$th difference. In $p_{2s-2}(j)$, the top $j^{s-1}$ coefficient is $1/(s-1)$, arising only from $(jt^2)^{s-1}/(s-1)$ in the logarithm. Consequently the coefficient of $m^s$ in $P_{m,2s-2}$ is $1/(s(s-1))$. The $s$th difference of $m^s$ is $s!$, giving $b_{g,s,2s-2} = (s-2)!$.

Let $\phi_d = \operatorname{rank} \operatorname{gr}_d G_\Gamma(\Sigma_g)$. Then $c_1 = \phi_1/(2g)$, and successively

$$c_s = \frac{\phi_{2s-2} - \sum_{j=1}^{s-1} c_j b_{g,j,2s-2}}{(s-2)!} \qquad (s \geq 2). \tag{34}$$

All larger clique blocks vanish in that degree, so the recursion is exact. It recovers every clique count; conversely the counts give all ranks by (6). This finishes Theorem 1.3. □

## 9. Examples and the boundary of the method

For genus one, the first six ranks are

$$\phi_1 = 2|V|, \quad \phi_2 = c_2, \quad \phi_3 = 2c_2, \quad \phi_4 = 3c_2 + c_3, \quad \phi_5 = 6c_2 + 4c_3, \quad \phi_6 = 9c_2 + 12c_3 + 2c_4.$$

These formulas follow by specializing the full product. In higher genus the singleton blocks have nonzero higher-degree contributions. For example,

$$c_2 = \phi_2 - |V| \left( \binom{2g}{2} - 1 \right),$$

so the genus-one identity $\phi_2 = c_2$ must not be used for $g > 1$.

An edgeless graph on $n$ vertices has group $(\pi_1\Sigma_g)^n$, and the formula reduces to $n$ singleton surface blocks. A complete graph recovers (4). For a disconnected graph both the group and its rank formula split over connected components. At a fixed genus, equality of clique polynomials gives equality of every lower-central rank. It does not assert an isomorphism of the corresponding groups or Lie rings.

The planar disk argument cannot be transferred verbatim to closed surfaces. Already for one retained vertex and $|V| \geq 2$, a homomorphism $s : \pi_1\Sigma_g \to P_V(\Sigma_g)$ with image in $J_i(V)$ and $q_{\{i\}} s = \mathrm{id}$ would embed the closed-surface group in the free group $\pi_1(\Sigma_g \setminus \{p_j : j \neq i\})$. Every subgroup of a free group is free, whereas the closed-surface group has $H_2 = \mathbb{Z}$ and is not free. Such a section does not exist. The surface integral proof instead uses the all-field root calculation and saturation. It proves strictness for the ordinary lower central series. We have not yet determined how far this strictness extends to verbal functors on surfaces.

The group theorem concerns the specified finite clique diagram and applies independently of asphericity. The linearity corollary concerns planar groups. The integral statements have independent proofs.

## References


[1] V. G. Bardakov and P. Bellingeri, *On residual properties of pure braid groups of closed surfaces*, Comm. Algebra **37** (2009), 1481–1490. doi:10.1080/00927870802068664; arXiv:0705.4201.

[2] V. G. Bardakov, R. Mikhailov, V. V. Vershinin and J. Wu, *Brunnian braids on surfaces*, Algebr. Geom. Topol. **12** (2012), 1607–1648. doi:10.2140/agt.2012.12.1607.

[3] G. M. Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), 178–218. doi:10.1016/0001-8708(78)90010-5.

[4] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124. doi:10.1090/S0002-9947-1948-0024908-8.

[5] D. C. Cohen and M. J. Falk, *On graphic arrangement groups*, Adv. Appl. Math. **126** (2021), 102130. doi:10.1016/j.aam.2020.102130; arXiv:1908.07664.

[6] B. Enriquez and V. V. Vershinin, *On the Lie algebras of surface pure braid groups*, preprint (2009). arXiv:0902.1963.

[7] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for homology via Galois theory*, Adv. Math. **217** (2008), 2231–2267. doi:10.1016/j.aim.2007.11.001; arXiv:math/0701815.

[8] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118. doi:10.7146/math.scand.a-10517.

[9] M. Falk and R. Randell, *The lower central series of a fiber-type arrangement*, Invent. Math. **82** (1985), 77–88. doi:10.1007/BF01394780.

[10] T. Kohno, *Série de Poincaré–Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985), 57–75. doi:10.1007/BF01394779.

[11] D. Krammer, *Braid groups are linear*, Ann. of Math. (2) **155** (2002), 131–156. doi:10.2307/3062152.

[12] H. Nakamura, N. Takao and R. Ueno, *Some stability properties of Teichmüller modular function fields with pro-l weight structures*, Math. Ann. **302** (1995), 197–213.

[13] C. Reutenauer, *Free Lie algebras*, London Math. Soc. Monographs, New Series, vol. 7, Clarendon Press, Oxford, 1993.