RESIDUE PAIRINGS AT THE CRITICAL KOSZUL DEGREE

A PARITY OBSTRUCTION TO GENERIC VANISHING, THE NINE-DIMENSIONAL HILBERT FUNCTION, AND PFAFFIAN DEGENERACY

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Abstract. Let V be an n-dimensional complex vector space and let $K \subseteq \bigwedge^2 V$ have dimension $2n-2$. We construct a residue pairing whose cokernel is the critical Koszul component $W_{n-4}(V,K)$ whenever the Plücker linear system K is base-point-free. The pairing is symmetric for even n and alternating for odd n. Its order is odd precisely when $n = 2^r + 1$, $r \geq 2$, and this proves critical nonvanishing for every K in those dimensions, including systems with base points. Thus the generic vanishing assertion in Conjecture 1.7 of Aprodu–Farkas–Raicu–Suciu fails in infinitely many dimensions, and their question about universal nonvanishing in the same dimensions has an affirmative answer. A deterministic finite-field minor certificate proves that the generic defect in dimension nine is exactly one and determines the complete Hilbert function. In the even-order alternating case, a relative version of the construction gives a global square root of the universal determinant, provided that determinant is not identically zero. We also describe the local Pfaffian scheme in the odd-order case and deduce nonvanishing of symmetric powers of kernel bundles on $K3$ surfaces.

1. The critical question and the main results

Let V be a complex vector space of dimension n and set $S = \mathrm{Sym}(V)$. For $q \geq 0$, put

$$W_q(V,K) = \operatorname{coker}\left\{K \otimes \mathrm{Sym}^q V \xrightarrow{\delta_2} B_q\right\}, \qquad B_q = \ker\left\{V \otimes \mathrm{Sym}^{q+1} V \xrightarrow{\delta_1} \mathrm{Sym}^{q+2} V\right\}, \tag{1.1}$$

where δ_1 is multiplication and

$$\delta_2((u \wedge v) \otimes a) = v \otimes ua - u \otimes va.$$

The direct sum $W(V,K) = \bigoplus_{q \geq 0} W_q(V,K)$ is a graded S-module. Exactness of the full Koszul complex gives the equivalent presentation

$$W_q(V,K) = \operatorname{coker}\left\{\bigwedge^3 V \otimes \mathrm{Sym}^{q-1} V \longrightarrow (\bigwedge^2 V/K) \otimes \mathrm{Sym}^q V\right\}, \tag{1.2}$$

with zero source when $q = 0$.

The resonance variety is

$$\mathcal{R}(V,K) = \{a \in V^* : \text{there is } b \in V^* \text{ with } 0 \neq a \wedge b \in K^\perp\} \cup \{0\}.$$

Let $G = \mathrm{Gr}_2(V^*)$ denote the Grassmannian of two-dimensional quotients of V, and let Q be its universal quotient. For $L = \det Q = \mathcal{O}_G(1)$, the identification $H^0(G,L) = \bigwedge^2 V$ makes K a linear system. It has no base point exactly when $\mathcal{R}(V,K) = \{0\}$.

We consider the second critical value of the number of relations,

$$\dim K = m = 2n-2, \qquad q = n-4, \qquad N = \dim \mathrm{Sym}^{n-4} V = \binom{2n-5}{n-4}. \tag{1.3}$$

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Throughout the paper $n \geq 5$. At this degree the map in (1.1) is square: $\dim B_q = mN$.

The source problem is the generic vanishing assertion in [1, Conjecture 1.7]: for $n \geq 6$ a general m-plane K should have $W_{n-4}(V, K) = 0$. The computations in [1, Section 8.2] suggested a one-dimensional defect for $n = 9$ and asked whether nonvanishing is universal for $n = 2^r + 1$. The following theorem explains this phenomenon by a pairing.

Theorem 1.1 (Critical residue pairing)**.** *Suppose K is base-point-free on G. There is a bilinear form b on $A = \mathrm{Sym}^{n-4} V$ such that*

$$b(v, u) = (-1)^{n-4} b(u, v), \qquad W_{n-4}(V, K) \cong \mathrm{coker}\{b : A \longrightarrow A^*\}.$$

In particular, when n is odd,

$$\dim W_{n-4}(V, K) \equiv N \pmod 2.$$

Theorem 1.2 (Universal nonvanishing)**.** *If $n = 2^r + 1$ with $r \geq 2$, then*

$$W_{n-4}(V, K) \neq 0$$

for every subspace $K \subseteq \bigwedge^2 V$ with $\dim K \leq 2n - 2$. For odd $n \geq 5$, the integer N in (1.3) is odd if and only if n has this form.

Theorem 1.3 (The generic module in dimension nine)**.** *For a general 16-dimensional subspace $K \subseteq \bigwedge^2 \mathbb{C}^9$,*

$$\mathrm{Hilb}_{W(V,K)}(t) = 20 + 96t + 270t^2 + 528t^3 + 660t^4 + t^5.$$

In particular, $\dim W_5(V, K) = 1$, $W_q(V, K) = 0$ for $q \geq 6$, and $\mathrm{reg}\, W(V, K) = 5$.

The proof of Theorem 1.3 contains a finite, exact computation: a fixed integer relation space, reduction modulo 32003, and a specified nonzero minor. The program and the minor indices are included as supplements, so this step is reproducible independently of any random choice. The inference from that finite calculation to a generic statement over $\mathbb{C}$ is proved in Section 5.

The pairing also controls the universal determinant. Let $X = \mathrm{Gr}_m(\bigwedge^2 V)$, let $\mathcal{S}$ be its universal subbundle, and put $\mathcal{O}_X(1) = \det \mathcal{S}^*$. The universal critical map is

$$\mathcal{D} : \mathcal{S} \otimes A \longrightarrow B_q \otimes \mathcal{O}_X. \tag{1.4}$$

After fixing the constant determinant factors, its determinant is a section of $\mathcal{O}_X(N)$.

Theorem 1.4 (The determinant and local Pfaffians)**.** *Assume n is odd.*

(1) *If N is even and $\det \mathcal{D}$ is not the zero section, there are an effective Cartier divisor E and a section $P \in H^0(X, \mathcal{O}_X(N/2))$ such that*

$$\mathrm{div}(\det \mathcal{D}) = 2E, \qquad \mathrm{div}(P) = E, \qquad \det \mathcal{D} = cP^2$$

for some $c \in \mathbb{C}^*$. The section P is unique up to nonzero scalar. For $n = 7$ the nonzero-determinant hypothesis holds and P has Plücker degree 42.

(2) *If N is odd, then $\det \mathcal{D} = 0$. On the base-point-free open set, the map is locally equivalent to $\mathrm{diag}(I_{(m-1)N}, b)$ with b an alternating $N \times N$ matrix of regular functions. If J is the ideal of its submaximal Pfaffians, then*

$$\mathrm{Fitt}_1(\mathrm{coker}\, \mathcal{D}) = J^2$$

on that chart. Its zero set is the locus with critical defect at least three.

We have not yet determined when E is reduced or irreducible, or when the local matrices assemble into one global scalar-valued alternating matrix. The Pfaffian principle for self-dual complexes is classical; see [3, Theorem 3.3 and Corollary 3.4]. Here it is realized directly by the critical Koszul linear system, and the global square is obtained from divisor multiplicities and the Picard group of X.

2. Grassmannian realization and the cohomology calculation

Set

$$D = \dim G = 2n-4, \qquad s = D+1 = 2n-3, \qquad q = n-4,$$

and define bundles and spaces

$$\mathcal{F} = \operatorname{Sym}^q Q, \qquad \mathcal{E} = \mathcal{F} \otimes L, \qquad A = H^0(G,\mathcal{F}), \qquad B = H^0(G,\mathcal{E}).$$

The Borel–Weil identifications are

$$A = \operatorname{Sym}^q V, \qquad B = \mathbb{S}_{(q+1,1)}V = B_q.$$

Here $\mathbb{S}_\lambda$ denotes the Schur functor of highest weight λ. Multiplication of sections

$$\bigwedge^2 V \otimes A = H^0(G,L) \otimes H^0(G,\mathcal{F}) \longrightarrow H^0(G,\mathcal{E})$$

agrees with δ_2 up to a nonzero constant. Indeed, the target $\mathbb{S}_{(q+1,1)}V$ occurs with multiplicity one in the source, and both maps are equivariant and nonzero. Nonvanishing of the Koszul map is seen on $(v_1 \wedge v_2) \otimes v_1^q$; nonvanishing of section multiplication follows by evaluation at a quotient where the two sections are nonzero. We fix compatible normalizations, obtaining

$$W_q(V,K) = B/(K \cdot A). \tag{2.1}$$

The elementary dimension formula

$$\dim B_q = n\binom{n+q}{q+1} - \binom{n+q+1}{q+2} = (q+1)\binom{n+q}{q+2}$$

gives $\dim B = mN$ at $q = n-4$.

The canonical bundle, with its constant equivariant factor retained, is

$$\omega_G \cong L^{-n} \otimes (\det V)^2.$$

Choose a volume form for V and hence an identification $\omega_G \cong L^{-n}$. Changing it rescales every scalar pairing by one nonzero constant and has no effect on rank or parity.

Lemma 2.1. *For $0 \le i \le s$, the only nonzero cohomology groups of $\mathcal{E} \otimes L^{-i}$ are*

$$H^0(G,\mathcal{E}) = B, \qquad H^0(G,\mathcal{E}(-1)) = A, \qquad H^D(G,\mathcal{E}(-s)) = A^*.$$

Proof. We apply the Borel–Weil–Bott rule in its Grassmannian form [4, Corollary 4.1.9]. For a bundle of weight $(a,b,0,\dots,0)$, add $\rho = (n-1,n-2,\dots,0)$. Repeated entries mean all cohomology vanishes; otherwise there is exactly one nonzero cohomology group, in the number of inversions of the resulting sequence.

The weight of $\mathcal{E}(-i)$ is $(q+1-i, 1-i, 0, \dots, 0)$, so the sequence to test is

$$(2n-4-i,\ n-1-i,\ n-3, n-4, \dots, 0). \tag{2.2}$$

For $i = 0, 1$ it is strictly decreasing, yielding the stated groups in degree zero. If $2 \le i \le n-1$, the second entry belongs to the tail $\{0,\dots,n-3\}$, so there is a repetition. If $n \le i \le D$, the first entry belongs to $\{0,\dots,n-4\}$, so there is again a repetition. At $i = s$ the first two entries are -1 and $2-n$, both smaller than all tail entries, and there are $2(n-2) = D$ inversions.

For a rank-two bundle, $\mathcal{F}^* \cong \mathcal{F} \otimes L^{-q}$. Since $q + n = D$,

$$\mathcal{E}(-s) = \mathcal{F} \otimes L^{1-s} = \mathcal{F} \otimes L^{-D} \cong \mathcal{F}^* \otimes \omega_G. \tag{2.3}$$

Serre duality identifies its top cohomology with A^*. □

Lemma 2.2. *Every base-point-free m-dimensional system K contains a base-point-free subspace K_0 of dimension $s = m-1$.*

Proof. The evaluation morphism $\phi_K : G \to \mathbf{P}(K^*)$ has closed image of dimension at most D, whereas $\dim \mathbf{P}(K^*) = D+1$. Choose $[y]$ outside that image and set $K_0 = \ker(y : K \to \mathbb{C})$. If x were a base point of K_0, then its nonzero evaluation functional on K would have kernel containing K_0. The two hyperplanes would be equal, giving $\phi_K(x) = [y]$, a contradiction. □

Proposition 2.3. *Let $K_0 \subseteq H^0(G, L)$ be base-point-free of dimension s. There is a connecting isomorphism*

$$B/(K_0 \cdot A) \cong \det K_0 \otimes H^D(G, \mathcal{E}(-s)) \cong \det K_0 \otimes A^*. \tag{2.4}$$

In particular, $K_0 \otimes A \to B$ is injective.

Proof. The evaluation $K_0 \otimes L^{-1} \to \mathcal{O}_G$ is surjective. Its Koszul complex, tensored with $\mathcal{E}$, is therefore exact:

$$0 \longrightarrow \bigwedge^s K_0 \otimes \mathcal{E}(-s) \longrightarrow \cdots \longrightarrow K_0 \otimes \mathcal{E}(-1) \longrightarrow \mathcal{E} \longrightarrow 0. \tag{2.5}$$

Index it from $-s$ to 0. By Lemma 2.1, its first hypercohomology page has only three nonzero positions:

$$(-s, D) : \det K_0 \otimes A^*, \quad (-1, 0) : K_0 \otimes A, \quad (0, 0) : B.$$

The kernel at $(-1, 0)$ cannot be killed by any later differential and must be zero, since the total hypercohomology of an exact complex is zero. The only remaining possible differential is d_{D+1} from $(-s, D)$ to the cokernel at $(0, 0)$, and it must be an isomorphism. Its inverse is the connecting map in (2.4). □

3. The residue formula and its symmetry

Choose an ordered basis $k_0, \ldots, k_D$ of K_0. It fixes the constant factor $\det K_0$ in (2.4). The affine open sets $U_i = \{k_i \neq 0\} \subseteq G$ cover G. They and their intersections are affine, being principal opens in the Plücker embedding. Hence the associated Čech complex computes coherent cohomology.

For $t \in B$, the residue form of the connecting class is

$$\rho(t) = \left[\frac{t}{k_0 \cdots k_D}\right] \in H^D(G, \mathcal{E}(-s)). \tag{3.1}$$

For completeness, this formula can be obtained by lifting t on U_i to te_i/k_i in $K_0 \otimes \mathcal{E}(-1)$, where e_i is the basis vector associated with k_i. On an intersection indexed by $i_0 < \cdots < i_p$, a further lift is

$$\frac{t\, e_{i_0} \wedge \cdots \wedge e_{i_p}}{k_{i_0} \cdots k_{i_p}}.$$

Koszul contraction gives the Čech boundary of the preceding lift: each k_{i_j} cancels precisely its own denominator. At the last stage the coefficient of $e_0 \wedge \cdots \wedge e_D$ is (3.1), up to one overall convention-dependent sign. If $t = k_i a$, cancellation of k_i makes the top cochain a boundary, so ρ vanishes on $K_0 \cdot A$. Proposition 2.3 shows that the resulting map on the quotient is an isomorphism.

The wedge pairing on Q induces a perfect bundle pairing

$$\langle\ ,\ \rangle : \operatorname{Sym}^q Q \otimes \operatorname{Sym}^q Q \longrightarrow L^q. \tag{3.2}$$

One explicit normalization is

$$\langle u_1 \cdots u_q, v_1 \cdots v_q \rangle = \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} \prod_{j=1}^{q} (u_j \wedge v_{\sigma(j)}).$$

It is symmetric in the variables within each input and hence descends to the two symmetric powers. Exchanging the two inputs gives

$$\langle v, u \rangle = (-1)^q \langle u, v \rangle. \tag{3.3}$$

In a local basis x, y of Q, the monomials $x^{q-i}y^i$ pair only with $x^i y^{q-i}$, with nonzero value $(-1)^i\binom{q}{i}^{-1}(x\wedge y)^q$. Thus the pairing is nondegenerate and realizes $\mathcal{F}\cong\mathcal{F}^*\otimes L^q$.

Under (2.3), the perfect pairing on (2.4) is consequently

$$B/(K_0\cdot A)\times A\longrightarrow\mathbb{C},\qquad([t],a)\longmapsto\operatorname{Tr}_G\left[\frac{\langle t,a\rangle}{k_0\cdots k_D}\right]. \tag{3.4}$$

Here the numerator has values in L^{q+1}, and $q+1-s=-n$, so the fraction is a top Čech cochain with values in ω_G. The map $\operatorname{Tr}_G: H^D(G,\omega_G)\to\mathbb{C}$ is the Serre trace.

Proof of Theorem 1.1. Choose K_0 as in Lemma 2.2 and write $K=K_0\oplus\mathbb{C}\ell$. The additional relations are the map

$$\mu_\ell: A\longrightarrow B/(K_0\cdot A),\qquad u\longmapsto[\ell u].$$

Compose it with (3.4). The resulting bilinear form is

$$b_\ell(u,v)=\operatorname{Tr}_G\left[\frac{\ell\,\langle u,v\rangle}{k_0\cdots k_D}\right]. \tag{3.5}$$

Since u, v are global sections in cohomological degree zero, their exchange introduces no additional Čech or Koszul sign. Equation (3.3) gives $b_\ell(v,u)=(-1)^q b_\ell(u,v)$. The quotient presentation (2.1) gives

$$W_q(V,K)=\operatorname{coker}\mu_\ell\cong\operatorname{coker}b_\ell.$$

When n is odd, so is q. An alternating matrix over $\mathbb{C}$ has even rank, yielding the asserted congruence. □

4. Binary parity and all relation spaces

Lemma 4.1. *If K has a base point on G, then $\dim W_q(V,K)\geq q+1$ for every $q\geq 0$.*

Proof. A base point corresponds to a quotient $\pi: V\twoheadrightarrow U$ with $\dim U=2$ and $(\bigwedge^2\pi)(K)=0$. By (1.2), the quotient map induces a surjection

$$W_q(V,K)\twoheadrightarrow W_q(U,0)=\bigwedge^2 U\otimes\operatorname{Sym}^q U.$$

Indeed, the ambient tensor map in that presentation is surjective and sends the relations into the target relations. The latter relations are zero because $\bigwedge^3 U=0$. The target has dimension $q+1$. □

Lemma 4.2. *For positive odd q, $\binom{2q+3}{q}$ is odd if and only if $q=2^r-3$ for some $r\geq 2$.*

Proof. In characteristic two, $(1+t)^a=\prod_{j:a_j=1}(1+t^{2^j})$, where a_j are the binary digits of a. Thus a binomial coefficient $\binom{a}{b}$ is odd exactly when every nonzero binary digit of b is a nonzero digit of a. Applied to $a=q+(q+3)$, this is equivalent to the addition of q and $q+3$ having no carry, or

$$q\,\&\,(q+3)=0,$$

where & is bitwise intersection. If $q\equiv 3 \pmod 4$, both q and $q+3$ have their second binary digit equal to one. Thus $q=4a+1$. The condition becomes $a\,\&\,(a+1)=0$.

If a has t consecutive trailing ones, adding one changes these to zeros, changes the next zero to one, and leaves every higher digit unchanged. Disjointness therefore holds exactly when all the higher digits of a vanish, that is, $a=2^t-1$. This gives $q=2^{t+2}-3$ and proves the lemma, including $a=0$. □

Proof of Theorem 1.2. For odd n, put $q=n-4$. Then $N=\binom{2q+3}{q}$, so Lemma 4.2 gives the stated parity classification. At $n=2^r+1$, a base-point-free K of dimension $2n-2$ yields an odd-order alternating form and hence a nonzero cokernel by Theorem 1.1. A K with a base point has nonzero cokernel by Lemma 4.1. If $\dim K<2n-2$, extend it to K' of dimension $2n-2$. The natural surjection $W_q(V,K)\twoheadrightarrow W_q(V,K')$ proves the same nonvanishing. □

Thus $n = 9, 17, 33, 65, \dots$ contradict the universal generic vanishing assertion of [1, Conjecture 1.7]. The case $n = 5$ recovers the earlier nonvanishing statement [1, Proposition 8.1]. When N is even, alternating symmetry alone does not decide whether the critical map is generically invertible; when n is even the pairing is symmetric and supplies no parity obstruction. Nor does this argument identify the present Koszul group with a Prym-canonical syzygy group.

5. A deterministic rank certificate in dimensions seven and nine

The only computational input is made precise in this section. All matrices start from integer coefficients. Reduction modulo one prime is used solely to certify that certain integer minors are nonzero.

5.1. **The fixed relations.** Let $n \in \{7, 9\}$ and let $V_{\mathbb{Z}}$ have basis $e_0, \dots, e_{n-1}$. Define

$$F(e_i) = (n-1-i)e_{i+1} \quad (0 \le i < n-1), \qquad F(e_{n-1}) = 0,$$

and extend F to $\bigwedge^2 V_{\mathbb{Z}}$ as a derivation. Set

$$k_j = F^j(e_0 \wedge e_1), \quad 0 \le j \le 2n-4, \qquad K_0 = \operatorname{span}_{\mathbb{C}}\{k_j\}.$$

These are the weight vectors of the highest summand $\operatorname{Sym}^{2n-4}\mathbb{C}^2 \subseteq \bigwedge^2 \operatorname{Sym}^{n-1}\mathbb{C}^2$. In particular, they are nonzero and have distinct weights $2n-4-2j$, so $\dim K_0 = 2n-3$. These facts can also be checked directly from the recursion defining the k_j.

For $n = 9$ put $\ell = \sum_{i<j} c_{ij} e_i \wedge e_j$, with the following upper-triangular coefficient array; blank entries are not used:

$i\backslash j$	1	2	3	4	5	6	7	8
0	67	−89	5	28	17	−57	70	−86
1		−54	−26	33	−8	−27	−47	−61
2			87	40	27	−99	57	16
3				−6	1	−10	89	75
4					83	94	5	59
5						−10	40	−43
6							2	−86
7								47

For $n = 7$, retain precisely the entries with $j \le 6$. Let $K = K_0 + \mathbb{C}\ell$. The certificate below proves that $\ell \notin K_0$ and therefore $\dim K = 2n-2$.

5.2. **Integral coordinates and quotient reduction.** Write $q = n-4$ and $N = \binom{2n-5}{n-4}$. For each monomial e^μ of total degree $q+2$, the coordinates above it in $V \otimes \operatorname{Sym}^{q+1} V$ are

$$e_j \otimes e^{\mu - \epsilon_j}, \qquad j \in \operatorname{supp}(\mu).$$

The multiplication map adds their coefficients. A vector in its kernel is consequently determined freely by all but the coordinate whose index is the least element of $\operatorname{supp}(\mu)$. Discarding that coordinate, for every μ, gives an integral basis of B_q. The inverse reinstates the discarded coefficient as the negative sum of the others. These coordinates remain valid over every field, with no denominators.

Give e_i weight $n-1-2i$. The map $K_0 \otimes \operatorname{Sym}^q V \to B_q$ preserves weights. Let C_w be the transpose of its matrix in weight w: each row is the coordinate vector of one relation. Its entries follow from

$$\delta_2((e_a \wedge e_b) \otimes e^\nu) = e_b \otimes e^{\nu+\epsilon_a} - e_a \otimes e^{\nu+\epsilon_b}.$$

An exact row reduction of C_w modulo $p = 32003$ gives pivot columns P_w and free columns T_w. The certificate verifies

$$\operatorname{rank} C_w = \text{number of rows of } C_w, \qquad \det C_w[:, P_w] \neq 0. \tag{5.1}$$

If the reduced row with pivot a is $e_a + \sum_{f\in T_w} r_{af}e_f$, define the quotient projection by

$$e_a \longmapsto -\sum_{f\in T_w} r_{af}e_f, \qquad e_f \longmapsto e_f.$$

Call this map π_w. The program verifies both

$$\pi_w C_w^{\mathsf{T}} = 0, \qquad \pi_w[:, T_w] = I. \tag{5.2}$$

Together with (5.1), these identities prove that π_w identifies the quotient by all K_0 relations. Their direct sum is a surjection $\pi : B_q \to \mathbb{F}_p^N$ with kernel precisely that relation space. Finally form

$$M = \pi \circ \delta_2|_{\ell\otimes \mathrm{Sym}^q V}.$$

It is an $N \times N$ matrix and

$$\operatorname{coker}\bigl(K \otimes \mathrm{Sym}^q V \to B_q\bigr) \cong \operatorname{coker} M \quad \text{over } \mathbb{F}_p. \tag{5.3}$$

Proposition 5.1 (Exact finite-field minors)**.** *For the relations just specified, the following data hold over* $\mathbb{F}_{32003}$*:*

n	N	$\operatorname{rank}(K_0 \otimes A \to B)$	$\dim B$	$\operatorname{rank} M$	*nonzero maximal minor of* M
7	84	924	1008	84	17710
9	1287	19305	20592	1286	27272

The last entry is a determinant modulo 32003*, of order* 84 *for* $n = 7$ *and* 1286 *for* $n = 9$*.*

Proof. This is the finite arithmetic part of the argument. The standalone supplement `01_critical_rank_certificate.py` constructs the matrices from the coefficients above, in the integral coordinates just described, using exact arithmetic in $\mathbb{F}_{32003}$. It checks that 32003 is prime by trial division up to its square root, verifies (5.1) and (5.2) in every weight block, and evaluates a nonzero maximal minor of M.

The recorded files `01_critical_rank_n7.json` and `01_critical_rank_n9.json` specify every block pivot set, its determinant, and the row and column indices of the final minor. Thus the displayed arithmetic assertions are reducible to specified finite determinant computations; the recorded rank is not used without its accompanying nonzero minors. The program depends only on Python and `python-flint`, and can be run as

```
python3 01_critical_rank_certificate.py 7 --output check7.json
python3 01_critical_rank_certificate.py 9 --output check9.json
```

Its monomial order is the recursive lexicographic listing of exponent tuples, with the first exponent increasing from zero; the target coordinates use that same order and increasing vector index. Weight blocks are ordered by decreasing weight. These conventions fix the matrices and the recorded minor indices completely. The reduced nine-dimensional matrix has SHA-256 checksum

`c8636bd88a93cc751a85aba3f2021995a149226810a4e417050d21655c1eb37d.`

The checksum identifies the reconstructed matrix and is not a substitute for the determinant verification. No random generator or cached matrix is used by the program. □

Lemma 5.2 (Lifting a rank certificate)**.** *Let* T *be an integer matrix. If its reduction modulo a prime has rank at least* r*, then* T *has rank at least* r *over* $\mathbb{C}$*.*

Proof. Some $r \times r$ minor has determinant nonzero modulo the prime. Its integer determinant is therefore nonzero and remains nonzero over $\mathbb{C}$. □

The quotient reductions above may use inverses modulo p, but this does not obstruct the lifting argument. Elementary row and column operations over $\mathbb{F}_p$ give

$$\operatorname{rank}\overline{\delta_2|_{K\otimes A}} = \operatorname{rank}\overline{\delta_2|_{K_0\otimes A}} + \operatorname{rank} M.$$

Consequently the original integer matrix has the stated finite-field rank; Lemma 5.2 applies to that original matrix, without lifting the inverses used in quotient coordinates.

5.3. **The complete generic Hilbert function.**

Proof of Theorem 1.3. Proposition 5.1 gives rank 20591 modulo 32003 for the full critical relation matrix in dimension nine. Its complex rank is at least 20591. Theorem 1.2 supplies the opposite bound $\dim W_5 \geq 1$, so the fixed complex instance has $\dim W_5 = 1$ and rank 20591. This also proves $\ell \notin K_0$, since otherwise the rank would be at most 19305.

The condition that the universal map on $\mathrm{Gr}_{16}(\bigwedge^2 \mathbb{C}^9)$ have rank at least 20591 is open. The fixed instance makes it nonempty, and the parameter space is irreducible. Universal nonvanishing bounds every rank by 20591. Hence the generic critical cokernel and kernel are both one-dimensional.

Let $J = \ker\{K \otimes S \to \bigoplus_{q\geq 0} B_q\}$ for such a general K. If $0 \neq a \in J_q$ with $q \leq 4$, multiplication gives an injection

$$\operatorname{Sym}^{5-q} V \longrightarrow J_5, \qquad h \longmapsto ha.$$

Indeed, $K \otimes S$ is free over the integral domain S, so a nonzero element is not killed by any nonzero polynomial. The source has dimension at least nine, contradicting $\dim J_5 = 1$. Thus $J_q = 0$ for $0 \leq q \leq 4$, and

$$\dim W_q = (q+1)\binom{9+q}{q+2} - 16\binom{8+q}{q} \qquad (0 \leq q \leq 4). \tag{5.4}$$

These values are $20, 96, 270, 528, 660$.

If this K had a base point, Lemma 4.1 would imply $\dim W_5 \geq 6$. Thus it is base-point-free. The effective vanishing theorem for Koszul modules with zero resonance states

$$\mathcal{R}(V,K) = \{0\} \quad \Longrightarrow \quad W_q(V,K) = 0 \quad (q \geq n-3); \tag{5.5}$$

see [2, Theorem 2.2], equivalently [1, equation (1.3)]. It gives vanishing for $q \geq 6$. This establishes the full series. A finite-length graded module has regularity equal to its last nonzero degree, so its regularity is five. □

For $n = 7$, Proposition 5.1 and Lemma 5.2 give a full-rank complex critical map. In particular, its universal determinant is not identically zero. No analogous assertion for all other values of n is inferred from the existence of the pairing.

6. Relative pairings and the global half-divisor

6.1. **The base-point boundary.** Put $M = \dim \bigwedge^2 V$ and let $Z \subseteq X$ be the parameter locus of systems with a base point on G. For $x \in G$, let $H_x \subseteq \bigwedge^2 V$ be the kernel of evaluation at x. The closed incidence variety

$$I = \{(K,x) \in X \times G : K \subseteq H_x\}$$

is a Grassmannian bundle over G with fiber $\mathrm{Gr}_m(\mathbb{C}^{M-1})$. Consequently

$$\dim I = D + m(M-1-m) = \dim X - 2.$$

Its proper projection has image Z, so

$$\operatorname{codim}_X Z \geq 2. \tag{6.1}$$

In particular, $U = X \setminus Z$ is nonempty and contains the generic point of every prime divisor of X.

6.2. **Algebraic reduction on the open set.** Fix $K \in U$, choose a base-point-free hyperplane $K_0 \subseteq K$, and choose a local algebraic frame of $\mathcal{S}$ whose first $m-1$ vectors span K_0 at K. On a sufficiently small Zariski neighborhood $T \subseteq U$, they form a subbundle $\mathcal{S}_0 \subseteq \mathcal{S}$ whose evaluation is still base-point-free. This shrinking is legitimate because the existence of a base point is closed, by the same proper incidence construction. The last frame vector ℓ trivializes $\mathcal{S}/\mathcal{S}_0$.

Use the relative evaluation Koszul complex on $G \times T$. Every bundle on G in Lemma 2.1 is fixed, so the projection formula and cohomology base change turn the calculation of Proposition 2.3 into an exact sequence of bundles

$$0 \longrightarrow \mathcal{S}_0 \otimes A \longrightarrow B \otimes \mathcal{O}_T \longrightarrow \mathcal{C}_0 \longrightarrow 0, \qquad \mathcal{C}_0 \cong \det \mathcal{S}_0 \otimes A^*. \tag{6.2}$$

After a further local splitting and trivialization, elementary invertible changes of coordinates transform $\mathcal{D}$ to

$$\operatorname{diag}(I_{(m-1)N}, b). \tag{6.3}$$

The entries of b are regular functions because the construction is an algebraic bundle map. Relative Serre duality identifies it with the pairing (3.5), so $b^{\mathsf{T}} = (-1)^q b$. Before trivialization, it is an A-pairing with values in $\det \mathcal{S}_0 \otimes (\mathcal{S}/\mathcal{S}_0)^*$. This line factor is why the local construction should not be confused with a global scalar-valued alternating matrix on X.

Proof of Theorem 1.4(1). The determinant line of (1.4) is

$$\det(B \otimes \mathcal{O}_X) \otimes \det(\mathcal{S} \otimes A)^* \cong (\det \mathcal{S}^*)^N \otimes \det B \otimes (\det A^*)^m.$$

The last two factors are constant lines, so the chosen volumes identify it with $\mathcal{O}_X(N)$.

Suppose N is even and the determinant is nonzero. On the charts (6.3), its local equation is $u \det b = u \operatorname{Pf}(b)^2$ for a regular unit u. Every prime divisor Γ has its generic point in U by (6.1), and hence

$$\operatorname{ord}_\Gamma(\det \mathcal{D}) = 2 \operatorname{ord}_\Gamma(\operatorname{Pf} b).$$

The effective Weil divisor $E = \frac{1}{2} \operatorname{div}(\det \mathcal{D})$ is integral. Since X is smooth, it is Cartier. The standard Picard group computation for a Grassmannian gives $\operatorname{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$; therefore $\mathcal{O}_X(2E) = \mathcal{O}_X(N)$ implies $\mathcal{O}_X(E) = \mathcal{O}_X(N/2)$. The canonical section of $\mathcal{O}_X(E)$ becomes a section P of this last line bundle. The rational function $\det \mathcal{D}/P^2$ and its inverse have zero valuation at every prime divisor. Normality extends them to regular functions on X, and projectivity and connectedness make them constants. Thus $\det \mathcal{D} = cP^2$ with $c \neq 0$. The same argument proves uniqueness up to scalar. No local square root of the unit u was assumed.

For $n = 7$, $N = 84$ and the nonzero-determinant hypothesis was proved by Proposition 5.1. The half-degree is 42. □

6.3. **Odd order and the Pfaffian scheme.**

Lemma 6.1. *Let b be an alternating matrix of odd order $N = 2r+1$ over a commutative $\mathbb{C}$-algebra. Delete row and column i to form $b_{\widehat{i}}$, and put*

$$p_i = (-1)^{i-1} \operatorname{Pf}(b_{\widehat{i}}), \qquad p = (p_1, \dots, p_N)^{\mathsf{T}}.$$

Then $bp = 0$ and $\operatorname{adj}(b) = pp^{\mathsf{T}}$. Consequently the ideal of $(N-1)$-minors is $(p_1, \dots, p_N)^2$.

Proof. The identity $bp = 0$ is the Pfaffian expansion with one repeated row and column. The adjugate identity may be verified over the polynomial ring in the independent entries of a universal alternating matrix. On the dense locus of rank $N-1$, both its rows and its columns lie in the one-dimensional kernel spanned by p, since the adjugate is annihilated on both sides by b. Thus it has the form $c\,pp^{\mathsf{T}}$ over the fraction field. A diagonal cofactor is $\det b_{\widehat{i}} = p_i^2$; choosing a nonzero p_i proves $c = 1$. Both sides are polynomial, so equality holds everywhere and hence after any specialization. The adjugate entries are, up to signs, all $(N-1)$-minors; they are exactly the products $p_i p_j$. □

Proof of Theorem 1.4(2). Every odd-order alternating b has zero determinant, so (6.3) makes $\det \mathcal{D}$ vanish on the dense open set U and thus on X. At a complex point, alternating rank is even. Its corank is at least three exactly when every submaximal Pfaffian vanishes. Elementary invertible transformations and removal of an identity block preserve Fitting ideals; for the resulting N-generator presentation, Fitt_1 is the ideal of $(N-1)$-minors. Lemma 6.1 identifies that ideal with J^2. □

In particular, for $n = 9$ the generic corank-one assertion is an independent rank result, supplied by the certificate. It does not follow from odd order alone. The Pfaffian ideal and the Fitting ideal give the same zero set but generally different scheme structures.

Proposition 6.2 (The nine-dimensional base-point boundary)**.** *For $n = 9$, the base-point locus $Z \subseteq X$ is irreducible of codimension two. Its general member has exactly one base point, and that point is reduced. The closed set*

$$\Sigma = \{K : \dim W_5(V, K) \geq 3\}$$

contains Z. In particular Σ cannot have pure codimension three in X.

Proof. Here $M = 36$, $m = 16$, $D = 14$, and $\dim X = 320$. The incidence variety I is irreducible of dimension 318. If $x \neq y$, their evaluation hyperplanes are distinct, since the Plücker system embeds G. The double-point incidence therefore has dimension

$$2D + m(M - 2 - m) = 28 + 16 \cdot 18 = 316.$$

Its image in I has closure of dimension at most 316. Thus a general $(K, x) \in I$ has no other base point; the projection is generically one-to-one and $\dim Z = 318$.

The sections vanishing at a fixed x have first derivatives spanning $T_x^* G$, because L is very ample. A general 16-dimensional subspace of these sections has derivative rank 14. In the local ring at x, its ideal consequently generates the maximal ideal modulo its square, and Nakayama's lemma makes the ideal equal to the maximal ideal. Thus the unique general base point is reduced. Finally, Lemma 4.1 gives $\dim W_5 \geq 6$ on Z. □

Proposition 6.3 (A nontrivial jump inside the open set)**.** *There is a base-point-free 16-plane $K \subseteq \bigwedge^2 \mathbb{C}^9$ with $\dim W_5(V, K) \geq 13$. The scheme defined by $\mathrm{Fitt}_1(\operatorname{coker} \mathcal{D})$ is nonreduced at such a point.*

Proof. Take $V = \mathrm{Sym}^8 \mathbb{C}^2$, with $e_i = x^{8-i} y^i$, and take the highest summand $K_0 = \mathrm{Sym}^{14} \mathbb{C}^2$ used in the certificate. This system is base-point-free. To see this directly, identify the dual SL_2 representation V^* with $\mathrm{Sym}^8 \mathbb{C}^2$. The dual projection of $\bigwedge^2 V^*$ onto the dual highest summand is, up to nonzero scalar, the first transvectant

$$(f, g) \longmapsto f_x g_y - f_y g_x.$$

For homogeneous binary forms of the same positive degree, its vanishing implies that the derivative of the rational function f/g in an affine chart is zero; in characteristic zero f/g is constant. Thus no nonzero decomposable element lies in $K_0^{\perp}$, proving base-point-freeness.

Set $\ell = e_0 \wedge e_3 - 3 e_1 \wedge e_2$. The raising operator $E(e_i) = i e_{i-1}$ kills ℓ, which has weight 10. It is the highest vector of the $\mathrm{Sym}^{10} \mathbb{C}^2$ summand and is not in K_0. Thus $K = K_0 \oplus \mathbb{C}\ell$ is still base-point-free. Proposition 2.3 is equivariant for SL_2, and $\det K_0$ is the trivial representation, so the quotient is A^* as a representation. Multiplication by ℓ raises weight by 10. All weights of $A = \mathrm{Sym}^5(\mathrm{Sym}^8 \mathbb{C}^2)$, and of A^*, lie between -40 and 40. Thus all source vectors of weights $32, 34, 36, 38, 40$ are killed by the reduced map. Their multiplicities are $5, 3, 2, 1, 1$: respectively, these count partitions of $4, 3, 2, 1, 0$ into at most five parts, each at most eight. The kernel has dimension at least 12. Alternating parity makes its dimension odd, so it is at least 13.

Near this point, let J be the submaximal Pfaffian ideal. It is proper at the point. It is nonzero in the local ring, because the generic defect is one and X is irreducible. Nakayama's lemma gives

$J \neq J^2$: if $J = J^2$, then $J \subseteq \mathfrak{m}J$ and hence $J = \mathfrak{m}J$, forcing $J = 0$. Choose $a \in J \setminus J^2$. Its class in $\mathcal{O}/J^2$ is nonzero and has square zero. Since the Fitting ideal is J^2, the asserted scheme is nonreduced. □

These propositions explain why an expected codimension for a completely unrestricted alternating matrix cannot be applied to the whole parameter space here. We have not yet determined the codimension of $\Sigma \cap U$ or criteria for a global reduced Pfaffian jump divisor.

7. A kernel-bundle consequence on projective varieties

Let Y be a complex projective variety and $\mathcal{B}$ a globally generated vector bundle. Write $H = H^0(Y, \mathcal{B})$ and $n = \dim H$, and let

$$d : \bigwedge^2 H \longrightarrow H^0(Y, \bigwedge^2 \mathcal{B})$$

be the determinant map. Assume $d \neq 0$, and set

$$M_{\mathcal{B}} = \ker(H \otimes \mathcal{O}_Y \to \mathcal{B}), \qquad V = H^*, \qquad K = (\ker d)^{\perp} \subseteq \bigwedge^2 V.$$

The geometric Koszul-module theorem of [2, Theorem 4.3] gives the natural identification

$$W_q(V, K)^* \cong \ker\left\{ H^1(Y, \operatorname{Sym}^{q+2} M_{\mathcal{B}}) \longrightarrow \operatorname{Sym}^{q+2} H \otimes H^1(Y, \mathcal{O}_Y) \right\}. \tag{7.1}$$

We use precisely this statement as a published input.

Corollary 7.1. *Suppose $n = 2^r + 1$ with $r \geq 2$ and $\operatorname{rank} d \leq 2n - 2$. Then the kernel in (7.1), with $q = n - 4$, is nonzero. In particular, if $H^1(Y, \mathcal{O}_Y) = 0$, then*

$$H^1(Y, \operatorname{Sym}^{n-2} M_{\mathcal{B}}) \neq 0.$$

Proof. By definition, $\dim K = \operatorname{rank} d \leq 2n - 2$. Apply Theorem 1.2 and then (7.1). The final assertion follows when the target of that kernel map is zero. □

Corollary 7.2. *Let Y be a $K3$ surface and $\mathcal{B}$ a globally generated rank-two vector bundle with $h^0(Y, \mathcal{B}) = 9$ and $h^0(Y, \det \mathcal{B}) \leq 16$. Then*

$$H^1(Y, \operatorname{Sym}^7 M_{\mathcal{B}}) \neq 0.$$

More generally, let $a \geq 2$ and $r = 2^a - 1$. If a polarized $K3$ surface (Y, H_Y) has $H_Y^2 = 4r$ and a globally generated rank-two bundle $\mathcal{B}$ satisfies $\det \mathcal{B} = H_Y$ and $h^0(Y, \mathcal{B}) = r + 2$, then

$$H^1(Y, \operatorname{Sym}^r M_{\mathcal{B}}) \neq 0.$$

Proof. Global generation of a rank-two bundle makes d nonzero: at any point two global sections may be chosen to span its fiber. Also $H^1(Y, \mathcal{O}_Y) = 0$. In the first assertion $\operatorname{rank} d \leq 16 = 2n - 2$, so Corollary 7.1 applies. In the second assertion the ample polarization on a $K3$ surface has $h^0(Y, H_Y) = 2 + H_Y^2/2 = 2r + 2$. Thus $\operatorname{rank} d \leq 2r + 2 = 2(r + 2) - 2$ and $n = r + 2 = 2^a + 1$; $n - 2 = r$ gives the conclusion. □

In the test family of [1, Section 8.3 and Proposition 8.2], this excludes the proposed symmetric-power vanishing hypothesis at $r = 3, 7, 15, 31, \ldots$, whenever the displayed global-generation and section-count assumptions hold. The conditional proposition retains its stated scope. We have not yet established a comparison with the Prym–Green groups.

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