KOSZUL STRUCTURES OF ELLIPTIC CONFIGURATION SPACES

ANSWERS TO SUCIU'S PROBLEM 22.8, CONJECTURE 22.11,
REMARK 22.20, AND THE QUADRATIC-ALGEBRA QUESTION IN QUESTION 22.25

DENZEL ZHENG

Abstract. We determine the entire cohomological Koszul module of the configuration space of n points on an elliptic curve. An explicit straightening argument identifies it, with its polynomial action and determinant twist, as the section module of $\mathcal{O}(0,2)$ on $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. This proves Suciu's Conjecture 22.11 and yields the complete linear resolution, depth, projective dimension, and ordinary-square realization. For an arbitrary graph, the analogous module is expressed by the Stanley–Reisner ring of its clique complex modulo the sum of the vertex coordinates. The comparison with the actual group is governed by the triangle ideal, with precisely two minimal generators per triangle. We also prove that the quadratic algebra displayed in Question 22.25 is Koszul for every n, by a polynomial–exterior Segre realization and a quadratic Gröbner basis. The full cohomology algebra of the punctured elliptic configuration space, asserted there to be equivalent, is Koszul only for zero or one point. The two questions therefore have different answers. Throughout, cohomological holonomy invariants are distinguished from the invariants of the non-1-formal fundamental group.

1. Questions, objects, and conventions

Let E be a complex elliptic curve. For a finite simple graph Γ on $[n] = \{1, \dots, n\}$, write

$$X_\Gamma = \mathrm{Conf}(E, \Gamma) = \{(z_1, \dots, z_n) \in E^n : z_i \neq z_j \text{ for } ij \in E(\Gamma)\}.$$

The complete-graph case is $X_n = \mathrm{Conf}(E, n)$. We work over $\mathbb{C}$ unless a different field is explicitly specified. All internal gradings below place derived Lie generators in degree zero; Lie degree is internal degree plus two.

For a connected graded cohomology algebra H with finite-dimensional H^1, put $V = (H^1)^*$ and let

$$I = \ker(\Lambda^2 H^1 \longrightarrow H^2), \qquad K = I^\perp \subset \Lambda^2 V.$$

Its cohomological holonomy Lie algebra and Koszul module are

$$\mathfrak{h}(H) = \mathrm{Lie}(V)/(K), \tag{1}$$

$$\mathcal{B}_q(H) = \mathrm{coker}\left(\Lambda^3 V \otimes \mathrm{Sym}^{q-1} V \xrightarrow{\bar{\delta}_3} (\Lambda^2 V/K) \otimes \mathrm{Sym}^q V\right), \quad q \geq 0. \tag{2}$$

The source is zero when $q = 0$, and

$$\delta_3(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes (v_2 \wedge v_3) - v_2 \otimes (v_1 \wedge v_3) + v_3 \otimes (v_1 \wedge v_2)$$

with the polynomial factor understood. Jacobi, followed by abelianizing the derived Lie algebra, gives $\mathcal{B}(H) \cong \mathfrak{h}(H)'/\mathfrak{h}(H)''$. One may obtain the complete presentation directly from the polynomial Koszul complex: the free metabelian derived module is $\mathrm{coker}\,\delta_3$ on $\Lambda^2 V$, and imposing the quadratic

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Lie relations quotients by their polynomial span. This is the usual infinitesimal Alexander invariant [5, 6].

In the version of Suciu's survey revised on 11 August 2026, Problem 22.8 asks for the complete-configuration module structure and projective dimension; Conjecture 22.11 specifies its full representation structure [8]. Remark 22.20 asks for the graphical comparison kernel in terms of triangles. We answer these by one presentation and its quotients. The quadratic algebra of Question 22.25 is treated in Section 7. Its alleged equivalence with a full-cohomology Koszul question requires a correction, proved in Section 8.

2. The topological input: the degree-one cup kernel

Let $W = H^1(E;\mathbb{C})$, $U = \mathbb{C}^n$ with basis u_i, and $U_0 = \{\sum c_i u_i : \sum c_i = 0\}$. Thus $H^1(E^n) = W \otimes U$. Choose a basis α, β of W and write $a_i = \alpha \otimes u_i$, $b_i = \beta \otimes u_i$, normalizing $\alpha\beta$ to the point class.

Lemma 2.1. *For every graph Γ,*

$$H^1(E^n;\mathbb{C}) \xrightarrow{\cong} H^1(X_\Gamma;\mathbb{C}), \qquad I_\Gamma = \operatorname{span}\{(a_i - a_j) \wedge (b_i - b_j) : ij \in E(\Gamma)\}.$$

These displayed cup-kernel elements are linearly independent. For the complete graph,

$$I_n = \Lambda^2 W \otimes \operatorname{Sym}^2 U_0 \tag{3}$$

under the natural Cauchy decomposition of $\Lambda^2(W \otimes U)$.

Proof. Set $Y = E^n$ and $D = \bigcup_{ij\in E(\Gamma)} \Delta_{ij}$. By supported Poincaré–Alexander duality, $H^1_D(Y) = 0$ and $H^2_D(Y) = \bigoplus_{ij\in E(\Gamma)} \mathbb{C}[\Delta_{ij}]$. Indeed D has real dimension $2n-2$, and distinct irreducible divisors meet in real codimension at least four in Y, so its top homology is the direct sum of their fundamental classes. The support sequence gives

$$0 \to H^1(Y) \to H^1(X_\Gamma) \to \bigoplus_{ij\in E(\Gamma)} \mathbb{C}[\Delta_{ij}] \to H^2(Y) \to H^2(X_\Gamma).$$

The difference map $z \mapsto z_i - z_j$ pulls the point class back to $(a_i - a_j)(b_i - b_j)$. In a linear combination of these classes, the coefficient of $a_i \wedge b_j$, $i < j$, is the negative of the coefficient of that unique edge. This proves independence, the H^1 assertion, and the cup-kernel assertion, since $H^2(Y) = \Lambda^2 H^1(Y)$.

For the complete graph let $f_i = u_i - u_n$. The roots of edges (i, n) give f_i^2, and the other roots give $(f_i - f_j)^2$. As two is invertible, their span is all $\operatorname{Sym}^2 U_0$. The skew W factor is $\Lambda^2 W$, proving (3). We have not asserted that degree-one products span the whole of $H^2(X_\Gamma)$. □

Put $A = W^*$, $B = U_0^*$, and $r = n - 1$. There is a two-dimensional central subspace $Z = W^* \otimes \operatorname{Ann}(U_0)$ of $V = H^1(X_n)^*$. Its exterior products with V are in K, so Z is central in the holonomy Lie algebra and acts by zero on its derived module. Modulo Z the vector space is $A \otimes B$, and

$$K_0 = \operatorname{Sym}^2 A \otimes \Lambda^2 B \subset \Lambda^2(A \otimes B). \tag{4}$$

The linear groups below act on this quadratic data, and hence on its holonomy and module; no action on the full topological cohomology ring is needed.

3. Straightening the entire complete-configuration module

Theorem 3.1. *For $n \geq 2$ there is an isomorphism of graded modules, equivariant for $\mathrm{GL}(W) \times \mathrm{GL}(U_0)$,*

$$\mathcal{B}_q(H^*(X_n;\mathbb{C})) \cong \Lambda^2 A \otimes \operatorname{Sym}^q A \otimes \operatorname{Sym}^{q+2} B, \qquad q \geq 0. \tag{5}$$

The two variables in Z act by zero. The remaining action is

$$(a \otimes b)(\eta \otimes p \otimes h) = \eta \otimes (ap) \otimes (bh). \tag{6}$$

In particular this proves Conjecture 22.11 of [8]*, including the determinant factor and compatibility in every internal degree.*

Proof. Take a basis a, b of A and a basis $z_1, \dots, z_r$ of B. Write $x_i = a \otimes z_i$, $y_i = b \otimes z_i$. The natural map

$$(a_1 \otimes v_1) \wedge (a_2 \otimes v_2) \longmapsto (a_1 \wedge a_2) \otimes v_1 v_2$$

identifies $\Lambda^2(A \otimes B)/K_0$ with $\Lambda^2 A \otimes \operatorname{Sym}^2 B$. Its generators can be denoted $t_{ij} = [x_i, y_j] = [x_j, y_i] = t_{ji}$, with image $(a \wedge b) \otimes z_i z_j$. All same-row brackets vanish.

The full Koszul relations from (2) are

$$x_i t_{jk} = x_j t_{ik}, \qquad y_i t_{jk} = y_j t_{ik}. \tag{7}$$

To see completeness, separate a basis of $\Lambda^3(A \otimes B)$ into three x's, three y's, two x's and one y, and one x and two y's. The first two types map to zero. The latter two give exactly the displayed relations, with all choices of indices. Thus the displayed relations give a complete presentation.

Let $M_q = \Lambda^2 A \otimes \operatorname{Sym}^q A \otimes \operatorname{Sym}^{q+2} B$ with action (6). Sending t_{ij} to $(a \wedge b) \otimes z_i z_j$ gives a module map $\mathcal{B} \to M$ because (7) holds in M. It is surjective: in a monomial $a^s b^t z^\gamma$, where $s + t = q$ and $|\gamma| = q + 2$, distribute q of its z indices among s variables x and t variables y, leaving two indices for t_{ij}.

For injectivity, regard a source monomial as q colored positions, with s colored x and t colored y, followed by the two uncolored positions of t. Each position carries one z index. A relation in (7) exchanges the index at any colored position with an index at an uncolored position; symmetry of t exchanges its two positions. These transpositions are the edges of a connected graph on the $q + 2$ positions, and generate its full symmetric group. For example, a transposition between two positions is the product of transpositions along a path and then back along that path. Hence the source monomial depends only on s, t and the multiset γ of indices. Repeated indices do not cause an exception: the corresponding exchange is the identity. Each such multidegree has dimension at most one; its image is a nonzero monomial, so it has dimension exactly one. Distinct target multidegrees are independent. The map is therefore injective in all degrees. The map was defined by natural exterior and symmetric products, so its equivariance is independent of the chosen bases. □

4. The Segre geometry and complete resolution

Corollary 4.1. *Let $S = \operatorname{Sym} H^1(X_n; \mathbb{C})^*$, a polynomial ring on $2n$ variables. Then*

$$\dim \mathcal{B}_q = (q+1)\binom{n+q}{q+2}, \tag{8}$$

$$\operatorname{Hilb}(\mathcal{B}, t) = \frac{(1-t)^n - 1 + nt}{t^2(1-t)^n} = \frac{\sum_{j=0}^{n-2} (-1)^j \binom{n}{j+2} t^j}{(1-t)^n}, \tag{9}$$

$$\operatorname{depth}_S \mathcal{B} = 2, \qquad \operatorname{pd}_S \mathcal{B} = 2n - 2, \qquad \operatorname{reg}_S \mathcal{B} = 0. \tag{10}$$

The minimal S-resolution is 0-linear and its only nonzero graded Betti numbers are

$$\beta_{i,i} = \binom{2n}{i+2} - \binom{n}{i+2} - n\binom{n}{i+1}, \qquad 0 \le i \le 2n-2. \tag{11}$$

Binomial coefficients outside their usual range are zero.

Proof. The dimension follows from Theorem 3.1. If

$$F(t) = \sum_{q \ge 0} \binom{n+q}{q+2} t^q = t^{-2}\big((1-t)^{-(n-1)} - 1 - (n-1)t\big),$$

then $\operatorname{Hilb}(\mathcal{B}, t) = (tF(t))'$, giving (9).

Discard the two annihilating variables and the one-dimensional determinant factor. The remaining module M is the full section module of $\mathcal{O}(0,2)$ on $Y = \mathbb{P}^1 \times \mathbb{P}^{r-1}$ under the Segre embedding:

$$M_q = H^0(Y, \mathcal{O}(q, q+2)).$$

There are no sections for $q < 0$, so it is saturated and $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$. The standard section-module comparison identifies $H^{i+1}_{\mathfrak{m}}(M)_q$ with $H^i(Y, \mathcal{O}(q, q+2))$ for $i \geq 1$ [3, §3.6].

For $r \geq 2$, the cohomology of a line bundle on projective space is nonzero only in degree zero or the top degree. Künneth therefore gives one intermediate group:

$$H^2_{\mathfrak{m}}(M)_{-2} = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbb{P}^{r-1}, \mathcal{O}) = \mathbb{C}.$$

No other degree contributes to this group: the two conditions are $q \leq -2$ and $q + 2 \geq 0$. Mixed cohomology from $H^0(\mathbb{P}^1, \mathcal{O}(q))$ and top cohomology of the other factor is impossible. Top local cohomology has highest nonzero degree $q = -r - 2 = -n - 1$, from the product of both top cohomology groups. Thus depth is two and regularity, the maximum of $q + i$ for nonzero $H^i_{\mathfrak{m}}$, is zero. For $r = 1$, the module is the two-variable polynomial ring, so the same depth and regularity hold.

Adding variables that annihilate M does not change these invariants. Auslander–Buchsbaum gives projective dimension $2n - 2$ over S. Generation in degree zero and regularity zero force every generator in homological degree i of the minimal resolution to have degree i. Multiplying (9) by $(1-t)^{2n}$ and comparing alternating Betti polynomials yields

$$\beta_{i,i} = \sum_{j=0}^{n-2} \binom{n}{j+2} \binom{n}{i-j}.$$

Vandermonde, with the omitted terms $j = -2, -1$ removed, gives (11). □

Proposition 4.2. *Let*

$$R = \mathbb{C}[x_1, y_1, \ldots, x_r, y_r]/(x_i y_j - x_j y_i : i < j).$$

The module M is a faithful rank-one reflexive R-module, locally free away from the cone vertex. If $P = (x_1, \ldots, x_r) \subset R$, then

$$P^{(2)} = P^2, \qquad M \cong P^2(2) \tag{12}$$

after choosing bases of the determinant and first row. For $n \geq 3$ it is not free at the vertex. Its annihilator over S is exactly the two central variables together with the Segre minors.

Proof. The ring R is the semigroup ring of monomials $a^s b^t z^\gamma$ with $s + t = |\gamma|$. This semigroup is saturated in its group, so R is a normal domain. On the punctured cone the section module comes from a line bundle, hence is locally free of rank one. Its depth at the vertex is at least two by Corollary 4.1; elsewhere it is locally free. It is torsion-free by its monomial realization, satisfies S_2, and is therefore reflexive over the normal ring. Being nonzero and torsion-free over the domain R, it has zero annihilator and is faithful. This also proves the stated annihilator over S. When $r \geq 2$, its degree-zero minimal generators number $\binom{r+1}{2} > 1$, so it cannot be a free rank-one module at the vertex.

The ideal P is a height-one prime. In the monomial realization its associated valuation is the exponent of a: on a chart $y_j \neq 0$ its generator is x_j and $x_i = (x_j/y_j)y_i$. Its second symbolic power thus has the monomial basis with a exponent at least two. Every such monomial can be factored by two generators az_i, since its total z degree is at least two. Hence $P^{(2)} = P^2$. Multiplication by a^2 identifies M_q with $(P^2)_{q+2}$, proving (12). The chosen-row identification in (12) describes the graded module, while (5) gives its natural $\mathrm{GL}(A)$-equivariant description. □

5. Graphical modules and holonomy Chen ranks

The complete-configuration theorem has a useful quotient version.

Proposition 5.1. *Let* $\dim A = 2$*, let* $C \subset \operatorname{Sym}^2 B^*$*, and put* $I = \Lambda^2 A^* \otimes C \subset \Lambda^2(A \otimes B)^*$*,* $K = I^\perp$*. If* $J \subset \operatorname{Sym} B$ *is generated by the quadratic space* $C^\perp$*, then the Koszul module of this pair satisfies*

$$W_q(A \otimes B, K) = \Lambda^2 A \otimes \operatorname{Sym}^q A \otimes (\operatorname{Sym} B/J)_{q+2}. \tag{13}$$

The polynomial action is still (6)*.*

Proof. The relations contain K_0 of (4). Relative to the module of Theorem 3.1, the additional degree-zero relations are exactly $\Lambda^2 A \otimes C^\perp$. In degree q their polynomial span is $\Lambda^2 A \otimes \operatorname{Sym}^q A \otimes (\operatorname{Sym}^q B \cdot C^\perp)$. Indeed the Cartan multiplication $\operatorname{Sym}^q(A \otimes B) \to \operatorname{Sym}^q A \otimes \operatorname{Sym}^q B$ is surjective: a pair of monomials is obtained by arbitrarily pairing their q factors. Taking this precise quotient gives (13), in every degree and with the actual multiplication. □

Let $\Delta(\Gamma)$ be the clique complex, and introduce formal variables $z_1, \dots, z_n$ and $\ell = \sum_i z_i$. Their classes modulo ℓ are the coordinate functions restricted to U_0. Put

$$R_\Gamma = \mathbb{C}[\Delta(\Gamma)]/(\ell) = \mathbb{C}[z_1, \dots, z_n]/(\ell, z_i z_j : ij \notin E(\Gamma),\ i \neq j). \tag{14}$$

Theorem 5.2. *For every graph on* $n \geq 2$ *vertices,*

$$\mathcal{B}_q(H^*(X_\Gamma; \mathbb{C})) \cong \Lambda^2 A \otimes \operatorname{Sym}^q A \otimes (R_\Gamma)_{q+2}. \tag{15}$$

If $c_k(\Gamma)$ *is the number of* k*-vertex cliques, then the holonomy Chen ranks are*

$$\dim(\mathfrak{h}(H^*(X_\Gamma))'/\mathfrak{h}(H^*(X_\Gamma))'')_{q+2} = (q+1) \sum_{k \geq 2} c_k(\Gamma) \binom{q}{k-2}, \qquad q \geq 0. \tag{16}$$

Disconnected graphs and isolated vertices are allowed.

Proof. Lemma 2.1 gives $I_\Gamma = \Lambda^2 W \otimes C_\Gamma$, where $C_\Gamma = \operatorname{span}\{(u_i - u_j)^2 : ij \in E(\Gamma)\}$. Its perpendicular space in $\operatorname{Sym}^2 B$ is spanned by the restricted nonedge monomials $z_i z_j$. They vanish on every edge root. They are independent: if a linear combination equals ℓL for a linear polynomial L, comparison of z_i^2 coefficients forces all coefficients of L to vanish. Their number $\binom{n}{2} - |E(\Gamma)|$ is exactly the dimension of $C_\Gamma^\perp$. Proposition 5.1 proves the module formula.

The Stanley–Reisner ring embeds into the product of the polynomial rings on maximal faces. The element ℓ restricts nontrivially on every face, so multiplication by it is injective. Counting monomials by their exact support gives

$$\operatorname{Hilb}(\mathbb{C}[\Delta(\Gamma)], t) = \sum_{k \geq 0} c_k(\Gamma) \left(\frac{t}{1-t}\right)^k, \quad c_0 = 1,$$

and quotienting by ℓ multiplies by $1 - t$. In degree $q + 2 \geq 2$ the $k = 0, 1$ terms contribute nothing, while the kth term contributes $c_k \binom{q}{k-2}$. Tensoring with $\operatorname{Sym}^q A$, of dimension $q+1$, proves (16). □

For $n = 1$ the space is E, its derived holonomy module is zero, and all displayed derived formulas are interpreted as zero. In the complete-graph case (16) reduces to (8). The Segre depth, reflexivity and resolution results need not extend to arbitrary graphs: the ring (14) can have vertex-supported nilpotents.

6. The actual group and the entire triangle comparison kernel

We state the precise classical group input needed for the comparison. Berceanu and coauthors prove that $G_\Gamma = \pi_1(X_\Gamma)$ is filtered-formal and that its Malcev Lie algebra is the degree completion of a Lie algebra L_Γ [2, Proposition 3.4]. In genus one this algebra has degree-one generators x_i, y_i with the following complete relations:

$$t_{ij} := [x_i, y_j] = [x_j, y_i], \qquad i \neq j, \tag{17}$$

$$t_{ij} = 0, \qquad ij \notin E(\Gamma), \tag{18}$$

$$[x_i, x_j] = [y_i, y_j] = 0, \qquad i \neq j, \tag{19}$$

$$[y_i, x_i] = \sum_{j \neq i} t_{ij}, \tag{20}$$

$$[x_k, t_{ij}] = [y_k, t_{ij}] = 0, \qquad k \notin \{i, j\}. \tag{21}$$

The t_{ij} are names for degree-two brackets, not additional degree-one generators. The quadratic relations (17)–(20) alone present $\mathfrak{h}_\Gamma = \mathfrak{h}(H^*(X_\Gamma))$. To verify this directly, the natural quotient map sends $[x_i, y_j]$ to $\eta \otimes z_i z_j$, $\eta = a \wedge b$. Modulo nonedges and ℓ its degree-two target has dimension $|E(\Gamma)|$, equal to the cup-kernel dimension, and $-z_i^2 = \sum_{j \neq i} z_i z_j$ gives (20). Thus no quadratic relation is omitted. The two global sum variables are central and vanish on the derived module, as before.

Consequently the identity on degree-one generators gives a canonical surjection $\Xi_\Gamma : \mathfrak{h}_\Gamma \to L_\Gamma$. This is the model comparison in Remark 22.20 of [8].

Proposition 6.1. *Let $S = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$, and for $ij \in E(\Gamma)$ put*

$$I_{ij} = (x_k, y_k : k \notin \{i, j\}) + (x_i + x_j, y_i + y_j).$$

Then

$$L'_\Gamma / L''_\Gamma \cong \bigoplus_{ij \in E(\Gamma)} S/I_{ij}. \tag{22}$$

In particular the rational group Chen ranks satisfy

$$\theta_1(G_\Gamma) = 2n, \qquad \theta_{q+2}(G_\Gamma) = (q+1)|E(\Gamma)|, \quad q \geq 0.$$

Proof. All quadratic brackets are generated by the edge t_{ij}. Relation (21) kills the variables off the endpoints. Applying x_j to (20) and using Jacobi gives $[x_j, t_{ij}] = -[x_i, t_{ij}]$: all other summands have x_j off their endpoints, and $[x_j, [y_i, x_i]] = [t_{ij}, x_i]$. Similarly $[y_j, t_{ij}] = -[y_i, t_{ij}]$. Thus there is a module surjection from the right-hand side of (22), sending its edge generator to t_{ij}.

For a fixed edge $e = \{i, j\}$, $i < j$, let F be the free metabelian Lie algebra on X, Y. Its derived module is $\mathbb{C}[X, Y]c$, where $[X, Y] = c$, $[X, pc] = Xpc$, $[Y, pc] = Ypc$, and $[pc, qc] = 0$. These rules satisfy Jacobi because multiplication by X, Y commutes. Set

$$x_i \mapsto X, \quad x_j \mapsto -X, \quad y_i \mapsto Y, \quad y_j \mapsto -Y,$$

and send the remaining generators to zero. Every relation (17)–(21) is respected: t_{ij} maps to $-c$ and every other edge bracket to zero; the endpoint relation maps to $-c$ on both sides; outside actions vanish. Therefore this is a genuine Lie map. Its derived map detects just the eth polynomial summand, with nonzero generator image. All edge maps together give a left inverse of the preceding surjection, proving (22).

Filtered formality identifies $\mathrm{gr}(G_\Gamma) \otimes \mathbb{C}$ with L_Γ. For a finitely generated filtered-formal group, Suciu–Wang's comparison for derived quotients identifies

$$\mathrm{gr}(G_\Gamma / G''_\Gamma) \otimes \mathbb{C} \cong L_\Gamma / L''_\Gamma$$

[9, Theorem 8.3]. This input does not assume 1-formality. Each polynomial summand has $q+1$ dimensions in internal degree q, which proves the rank formula. Its complete-graph specialization is an already established formula [8, Corollary 22.2], not an additional new conjecture answer. □

Let $J_\triangle \subset R_\Gamma$ be the ideal generated by $z_i z_j z_k$ over all triangles $\{i,j,k\}$.

Theorem 6.2. *The comparison kernel is the following exact graded submodule under (15):*

$$\ker \mathcal{B}(\Xi_\Gamma)_q = \Lambda^2 A \otimes \mathrm{Sym}^q A \otimes (J_\triangle)_{q+2}. \tag{23}$$

It is generated in internal degree one, with exactly $2c_3(\Gamma)$ minimal generators. Its degree-q dimension is

$$(q+1)\sum_{k\geq 3} c_k(\Gamma)\binom{q}{k-2}. \tag{24}$$

Thus the comparison is an isomorphism exactly for triangle-free graphs.

Proof. If a degree-one generated Lie algebra $\mathfrak{p}$ is quotiented by a homogeneous relation space $T \subset \mathfrak{p}'$, the kernel on derived modules is the $\mathrm{Sym}(\mathfrak{p}_1)$-span of the images of T. Indeed the kernel is $(\langle T\rangle + \mathfrak{p}'')/\mathfrak{p}''$; adjoint actions from $\mathfrak{p}'$ vanish in this quotient, and the remaining degree-one adjoint actions commute. Apply this fact to the extra relations (21). Under (15) their images are

$$\eta \otimes a \otimes z_i z_j z_k, \qquad \eta \otimes b \otimes z_i z_j z_k.$$

If the triple is not a triangle its monomial contains a nonedge factor and is already zero. For a triangle these are precisely its two generators. Cartan multiplication is surjective in every degree, as in Proposition 5.1; hence their polynomial span is exactly (23).

We verify both the map and the absence of any saturation issue. Let $\Delta_1(\Gamma)$ be the one-dimensional simplicial complex with just the vertices and edges. Every larger clique contains a triangle, so

$$R_\Gamma/J_\triangle = \mathbb{C}[\Delta_1(\Gamma)]/(\ell).$$

The sum ℓ is a nonzerodivisor for the same face-restriction reason as before. Counting monomials gives

$$\mathrm{Hilb}(R_\Gamma/J_\triangle, t) = 1 + (n-1)t + |E(\Gamma)|\frac{t^2}{1-t}.$$

For $d \geq 2$ evaluate degree-d forms at the oriented edge roots $v_{ij} = u_i - u_j$. This is defined on the quotient: ℓ, nonedges and triangles vanish on each root. The monomial $z_i^{d-1} z_j$ has value -1 at v_{ij} and zero at every other oriented edge root. Thus the evaluation map is surjective, and the displayed Hilbert series makes it an isomorphism in every degree $d \geq 2$. Under the edge map used in Proposition 6.1, the derived-module map is precisely

$$\eta \otimes p \otimes h \longmapsto h(v_{ij})\eta \otimes p.$$

This follows on the bracket generators and respects multiplication because $z(v_{ij})$ is exactly the scalar by which $a \otimes z$ acts in that edge summand. Thus its kernel is the ideal in (23) in every relevant degree, even for disconnected graphs.

Finally the triangle monomials are independent in $(R_\Gamma)_3$. If a linear combination were ℓF_2 in the Stanley–Reisner ring, write F_2 as a combination of squares and edge monomials. Comparison of the coefficients of z_i^3 kills all squares; comparison of $z_i^2 z_j$ on each edge then kills every edge term. Thus $F_2 = 0$. The ideal has no lower-degree terms and is generated in degree three, so its c_3 basis elements yield exactly two minimal degree-one module generators each. Subtracting the edge-only Hilbert dimensions from (16) proves (24). □

The theorem answers the metabelian kernel question of Remark 22.20. It does not compute all higher Lie syzygies of $\ker \Xi_\Gamma$. The known 1-formality criterion [2, Propositions 3.6–3.7] is explained by these formulas: triangles already yield a discrepancy of $2c_3$ in Lie degree three, whereas for a triangle-free graph the cubic relations follow from the quadratic ones by Jacobi. For a cubic relation attached to edge ij and outside vertex k, one of ik, jk is absent; expressing t_{ij} by the appropriate cross-bracket makes both Jacobi terms zero.

7. The quadratic algebra of Question 22.25

The displayed algebra in [8, §22.9, equation (22.38)] is

$$Q_n = \Lambda(a_1, b_1, \dots, a_n, b_n)/((a_i - a_j)(b_i - b_j) : i < j). \tag{25}$$

It is the quadratic closure determined by the cup kernel, not the full cohomology ring. In this section the field k can be any field of characteristic different from two.

Theorem 7.1. *The algebra Q_n is Koszul for every $n \geq 1$. Its Hilbert series is*

$$\mathrm{Hilb}(Q_n, t) = (1+t)^n(1+nt).$$

Let $\mathfrak{h}_n = \mathfrak{h}(H^*(\mathrm{Conf}(E, n); \mathbb{Q}))$ be the cohomological holonomy Lie algebra. Its enveloping algebra consequently satisfies

$$\mathrm{Hilb}(U(\mathfrak{h}_n), t) = \frac{1}{(1-t)^n(1-nt)},$$

and

$$\dim(\mathfrak{h}_n)_1 = 2n, \qquad \dim(\mathfrak{h}_n)_k = \frac{1}{k}\sum_{d|k} \mu(d) n^{k/d}, \quad k \geq 2.$$

These latter ranks are holonomy ranks, not the actual elliptic pure braid group's LCS ranks.

Proof. Set $m = n-1$ and $p_i = a_i - a_n$, $q_i = b_i - b_n$. The change of generators is invertible and gives

$$Q_n = B_m \otimes_{\mathrm{gr}} \Lambda(a_n, b_n), \qquad B_m = \Lambda(p_1, q_1, \dots, p_m, q_m)/(p_i q_i,\ p_i q_j + p_j q_i : i < j).$$

The first tensor product is graded, so its odd generators anticommute. Let $U' = k^2$ with basis s, t and $W' = k^m$ with basis e_i. The *ordinary* diagonal Segre product

$$R_m = \mathrm{Sym}(U') \,\#\, \Lambda(W') = \bigoplus_{d \geq 0} \mathrm{Sym}^d(U') \otimes \Lambda^d(W')$$

has polynomial multiplication in the first factor and exterior multiplication in the second, with no additional tensor sign. The map $p_i \mapsto s \otimes e_i$, $q_i \mapsto t \otimes e_i$ respects all relations and exterior anticommutativity.

Use exterior lexicographic order $q_1 > p_1 > q_2 > p_2 > \cdots > q_m > p_m$. The relation with $i < j$ reads $q_i p_j = p_i q_j$, with leading monomial $q_i p_j$; the diagonal leading monomial is $q_i p_i$. A monomial avoiding them uses each index at most once, and, in its increasing list of indices, all p choices precede all q choices. Thus its standard form is

$$p_{i_1} \cdots p_{i_a} q_{i_{a+1}} \cdots q_{i_d}, \qquad i_1 < \cdots < i_d, \quad 0 \leq a \leq d.$$

Every monomial reduces to such a form: diagonal products kill a repeated index, and an inversion $q_i p_j$ is replaced by a strictly smaller monomial. Their images are exactly the independent basis elements $s^a t^{d-a} \otimes (e_{i_1} \wedge \cdots \wedge e_{i_d})$ of R_m. The forms therefore span and are independent, proving $B_m \cong R_m$. Independence of remainders also proves that the quadratic relations are a Gröbner basis in every degree. The quadratic PBW criterion for Koszulness now gives that B_m is Koszul. This is the classical quadratic-Gröbner mechanism; equivalently one can use preservation of Koszulness under Segre products [1, 7]. Tensoring with an exterior Koszul algebra preserves Koszulness, so Q_n is Koszul.

The standard forms give $\dim(B_m)_d = (d+1)\binom{m}{d}$. For $m \geq 1$,

$$\mathrm{Hilb}(B_m, t) = (1+t)^m + mt(1+t)^{m-1} = (1+t)^{m-1}(1+(m+1)t).$$

For $m = 0$ it is k. Multiplication by $(1+t)^2$ proves the Hilbert series for all n. The quadratic dual of Q_n is $U(\mathfrak{h}_n)$ by the cup-kernel definition. Koszul duality gives its displayed series. PBW compares it with $\prod_{k\geq 1}(1-t^k)^{-\dim(\mathfrak{h}_n)_k}$. The factor $(1-t)^{-n}$ contributes n additional degree-one dimensions; logarithmic Möbius inversion of $(1-nt)^{-1}$ gives the free-Lie Witt numbers, proving the formulas. □

Equality with a free Lie rank sequence in degrees at least two is not an assertion that $\mathfrak{h}_n$ is a direct product of that free Lie algebra and an abelian Lie algebra. Nor does the calculation establish an actual group LCS formula: for $n \geq 3$ the group is not 1-formal, and the cubic relations of Section 6 change its Lie algebra. Already for $n = 3$ the holonomy third rank is eight, whereas the actual group third rank is six [8, Corollary 22.7].

8. Why full cohomology has the opposite answer

Write $E^* = E \setminus \{0\}$. Question 22.25 states that Koszulness of (25) is equivalent to Koszulness of $H^*(\mathrm{Conf}(E^*, n-1); k)$. The quadratic algebra result above is affirmative, but the following topological calculation disproves that equivalence in the standard cohomological grading over $\mathbb{Q}$.

Theorem 8.1. *The rational cohomology algebra $H^*(\mathrm{Conf}(E^*, m); \mathbb{Q})$ is Koszul if and only if $m = 0$ or $m = 1$. For every $m \geq 2$ it is not generated in degree one.*

Proof. Start with $m = 2$. The space is the complement in E^2 of the three divisors $x = 0$, $y = 0$, and $x = y$. Their classes are

$$a_1 b_1, \quad a_2 b_2, \quad (a_1 - a_2)(b_1 - b_2),$$

which are independent. The same support sequence as in Lemma 2.1 gives $b_1 = 4$ and a cup-product image of dimension $\binom{4}{2} - 3 = 3$.

The Fadell–Neuwirth fibration [4] has base E^* and fiber $E \setminus \{0, x\}$. They retract onto graphs of ranks two and three, respectively. The Serre spectral sequence, including its possible local coefficient action, has only base columns zero and one and fiber rows zero and one. Hence cohomology vanishes above degree two. Its finite-type Euler characteristic is $(-1)(-2) = 2$. Thus $1 - 4 + b_2 = 2$ and $b_2 = 5$. The two dimensions of H^2 outside the degree-one cup image are indecomposable. In particular the full Hilbert series $1 + 4t + 5t^2$ differs from the based quadratic closure B_2, whose series is $1 + 4t + 3t^2$.

For every $m \geq 2$, the forgetting map $f_m : \mathrm{Conf}(E^*, m) \to \mathrm{Conf}(E^*, 2)$ has a continuous section. Choose a coordinate disk at 0, with a metric Euclidean in that disk, and choose $\epsilon > 0$ within its radius. For an input (x, y) put

$$\rho(x, y) = \tfrac{1}{10} \min\{\epsilon, \mathrm{dist}(x, 0), \mathrm{dist}(y, 0)\}.$$

Fix distinct constants $0 < c_3 < \cdots < c_m < 1$ and a radial ray in the disk, and add points at radii $c_j \rho(x, y)$. They are nonzero, mutually distinct, and closer to the puncture than either x or y. They vary continuously, and forgetting them recovers the input. Thus $s_m^* f_m^* = 1$ on cohomology, so s_m^* is a surjective graded algebra map onto $H^*(\mathrm{Conf}(E^*, 2); \mathbb{Q})$. A quotient of a degree-one generated algebra is degree-one generated, contradicting the calculation above. Therefore none of the algebras with $m \geq 2$ is degree-one generated. A connected Koszul algebra must be degree-one generated, so none is Koszul.

For $m = 0$ the algebra is $\mathbb{Q}$; for $m = 1$ it is $\mathbb{Q} \oplus \mathbb{Q}^2$ with the positive-degree product zero. Its reduced bar complex has zero differential and is diagonal, so it is Koszul. This proves the exact classification without any assumption of trivial fibration monodromy. □

9. The answers and their scope

The module isomorphism and resolution answer Problem 22.8 and Conjecture 22.11. The triangle ideal answers the metabelian comparison question of Remark 22.20, with all internal degrees and minimal generators. Question 22.25 receives an affirmative answer for its displayed quadratic algebra and a negative answer for the full-cohomology formulation when $n \geq 3$. The Segre/PBW criterion used in that answer is classical; the direct straightening makes its application explicit. The graph formula concerns cohomological holonomy until the separate filtered-formal comparison is invoked. These distinctions are essential to the development: quadratic closure, full cohomology, and the actual group have different information. The conclusions follow from the proofs above.

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