LIMITS OF LOCAL REGULARITY IN ENRICHED HOMOLOGY

NEGATIVE ANSWERS TO THE COHEN–MACAULAY SUPPORT QUESTION OF APRODU–FARKAS–RAICU–SAMMARTANO–SUCIU AND THE PARAMETER-DETERMINACY QUESTION IN FLØYSTAD'S PROBLEM 4

DENZEL ZHENG

Abstract. We give two negative answers concerning the information about enriched homology imposed by local combinatorial regularity. The support resonance loci of a Cohen–Macaulay simplicial complex need not propagate. A three-vertex path gives an explicit example, and a family of shellable flag triangulations of a two-dimensional ball gives examples in which both adjacent Koszul modules are nonzero. This answers the question in Section 5 of Aprodu–Farkas–Raicu–Sammartano–Suciu. Separately, the parameters of an (a, b)-design do not determine its reduced homology. Two neighborly twelve-vertex surfaces have identical parameters and identical face numbers, but distinct Betti numbers over every field of characteristic different from two. This answers the numerical question in Fløystad's Problem 4; the same mechanism gives infinitely many parameter sets. Both results are expressed through one squarefree face chain complex. Its multigraded strands detect homology of induced subcomplexes, whereas its generic ranks detect homology of the entire complex. The resulting module calculations identify both the failure of support propagation and the failure of homology determination.

1. Two questions about one homology module

Let Δ be a finite simplicial complex on its actual vertex set V, and let $\Bbbk$ be a field. Write

$$S = \Bbbk[x_v : v \in V], \qquad E = \bigwedge_{\Bbbk}(e_v : v \in V), \qquad A(\Delta) = E/(e_F : F \notin \Delta).$$

Every singleton is a face, and the empty face is included. Fix an ordering of V to orient the monomials e_F. The squarefree face chain complex is

$$C_i(\Delta) = \bigoplus_{\substack{F \in \Delta \\ |F|=i}} S(-\mathbf{1}_F)e_F, \qquad \partial(e_F) = \sum_{v \in F} (-1)^{\mathrm{pos}(v,F)-1} x_v e_{F\setminus\{v\}}. \tag{1.1}$$

Here $\mathbf{1}_F \in \mathbb{N}^V$ is the characteristic vector of F, and $\mathrm{pos}(v, F)$ is its position in the ordered face. The empty-face term is $C_0 = S$. For each pair of deleted vertices, the two orders of deletion have opposite signs, so $\partial^2 = 0$. Define

$$W_i(\Delta) = H_i(C_\bullet(\Delta)), \qquad \mathcal{R}_i(\Delta) = V(\mathrm{Ann}_S W_i(\Delta)) \subseteq \Bbbk^V. \tag{1.2}$$

The affine scheme is $\mathrm{Spec}(S/\operatorname{Ann}_S W_i)$; statements about its support can equally be read in $\operatorname{Spec} S$. Each W_i is finitely generated, since $C_\bullet$ is finite free and S is Noetherian; thus its scheme-theoretic support is $V(\mathrm{Ann}_S W_i)$. These conventions give $W_0 = \Bbbk$ and $\mathcal{R}_i = \varnothing$ if $W_i = 0$. The indexing by face cardinality is one greater than the reduced simplicial homology index. Up to this index convention, (1.1) is Fløystad's enriched chain complex [3]; it is also the higher Koszul complex used in [1].

Date: Research note / October 2026.

2020 *Mathematics Subject Classification.* Primary 13F55; Secondary 13D02, 05B05, 57M15.

Key words and phrases. enriched homology, higher Koszul module, support resonance, Cohen–Macaulay complex, simplicial design, neighborly surface.

The *jump resonance* locus is defined by

$$\mathcal{R}^i(\Delta) = \{a \in \Bbbk^V : H^i(A(\Delta), a \wedge -) \neq 0\}, \qquad a = \sum_{v \in V} a_v e_v. \tag{1.3}$$

The cochain complex in (1.3) is obtained by specializing the dual of (1.1). Specialization before homology need not agree with taking homology before specialization.

The first question occurs in [1, Section 5, between Examples 5.4 and 5.5]: do the support loci $\mathcal{R}_i(\Delta)$ propagate when Δ is Cohen–Macaulay? This asks whether $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots$ holds in that class. The subscript denotes the support of enriched homology in (1.2); the superscript in (1.3) denotes the specialized cohomology degree.

The second occurs in [3, Section 5, Problem 4]. Within the broader request to determine the homological behavior of (a, b)-designs, it asks whether the dimensions of $\widetilde{H}_i(\Delta; \Bbbk)$ depend only on the numerical parameters n, d, c, a, b. The definition of a design is recalled in Section 4. We answer this explicit parameter-determinacy question negatively.

Theorem 1.1. *The following statements hold.*

(1) *Over every field, there are shellable flag triangulations D_r of a two-dimensional ball, one for each $r \geq 4$, for which*

$$W_1(D_r) \neq 0, \qquad W_2(D_r) \neq 0, \qquad \mathcal{R}_1(D_r) \not\subseteq \mathcal{R}_2(D_r).$$

More precisely, in variables $t, v_1, \ldots, v_r, x$,

$$W_2(D_r) \cong S/(t)(-\mathbf{1}_{\{v_1,\ldots,v_r\}}),$$

and $\mathcal{R}_1(D_r)$ contains the coordinate plane on $\{t, x\}$.

(2) *For every field of characteristic different from two, there are two $(1,1)$-designs with identical parameters $(n, d, c, a, b) = (12, 3, 2, 1, 1)$ and identical f-vector $(12, 66, 44)$, but with reduced Betti pairs $(\widetilde{\beta}_1, \widetilde{\beta}_2)$ equal to $(12, 1)$ and $(11, 0)$. Their enriched modules have different generic ranks in degrees two and three.*

(3) *There are such pairs with parameters $(n, 3, 2, 1, 1)$ for every $n = 12s$, $s \geq 1$. If $h = (n-3)(n-4)/12$, their Betti pairs are $(2h, 1)$ and $(2h-1, 0)$.*

The ball examples determine the support of enriched homology through induced subcomplexes. The design examples determine its generic rank through the homology of the whole complex. In the latter examples the design condition is imposed on deleted links, as in Definition 4.1.

2. Multigraded strands, multiplication, and generic rank

For $U \subseteq V$, let Δ_U be the induced subcomplex. Reduced chains include the augmentation term in degree -1; in particular, $\widetilde{H}_{-1}(\{\varnothing\}; \Bbbk) = \Bbbk$.

Lemma 2.1. *For $\alpha \in \mathbb{N}^V$ with support U, there is a natural isomorphism*

$$[W_i(\Delta)]_\alpha \cong \widetilde{H}_{i-1}(\Delta_U; \Bbbk). \tag{2.1}$$

If $\alpha_v > 0$, multiplication by x_v identifies this strand with the strand at $\alpha + \mathbf{1}_v$. If $\alpha_v = 0$, that multiplication is the homology map induced by $\Delta_U \hookrightarrow \Delta_{U \cup \{v\}}$. Moreover,

$$\operatorname{rank}_S W_i(\Delta) = \dim_\Bbbk \widetilde{H}_{i-1}(\Delta; \Bbbk). \tag{2.2}$$

Proof. In degree α, the summand indexed by F in (1.1) is nonzero exactly when $F \subseteq U$. It then has the basis vector $x^{\alpha - \mathbf{1}_F} e_F$. The boundary of this vector is the alternating sum of the corresponding basis vectors indexed by $F \setminus \{v\}$. Thus the entire strand is the augmented simplicial chain complex of Δ_U, with its degree shifted by one. This proves (2.1). Multiplication by x_v takes every such basis vector to the identical normalized face in the next strand. It is the identity if the support stays fixed, and is the induced inclusion map if a vertex enters the support.

Let K be the fraction field of S. In $K \otimes_S C_\bullet$, replace e_F by

$$\widehat{e}_F = \left(\prod_{v \in F} x_v\right)^{-1} e_F.$$

Its boundary is the ordinary alternating face boundary. Hence the localized complex is the augmented chain complex of Δ over K. Localization is exact, and extending coefficients from $\Bbbk$ to K preserves the homology dimensions of this finite complex. This gives (2.2). □

The strand formula is classical in enriched homology and squarefree Koszul theory; see [3, 1]. Its multiplication maps identify the entire module in the examples below.

Lemma 2.2 (Coning)**.** *If $c * \Delta$ is the cone with a new vertex c, then, as a module over $S[x_c]$,*

$$W_i(c * \Delta) \cong W_i(\Delta) \otimes_{\Bbbk} \Bbbk[x_c]/(x_c). \tag{2.3}$$

Coning preserves the homological index of the enriched modules.

Proof. The faces of the cone are F and $\{c\} \cup F$, with $F \in \Delta$. Ordering c last identifies its face complex with the tensor product of $C_\bullet(\Delta)$ and the two-term complex

$$\Bbbk[x_c](-1) \xrightarrow{x_c} \Bbbk[x_c].$$

The latter has homology $\Bbbk[x_c]/(x_c)$ in degree zero only. The Künneth formula over the field $\Bbbk$, or the long exact sequence of the mapping cone of multiplication by x_c, gives (2.3). □

3. Cohen–Macaulayness does not imply support propagation

We use the following precise classical criterion. A finite pure complex Δ is Cohen–Macaulay over $\Bbbk$ if and only if, for every face F, including the empty face,

$$\widetilde{H}_j(\mathrm{lk}_\Delta F; \Bbbk) = 0 \quad \text{for } j < \dim \mathrm{lk}_\Delta F. \tag{3.1}$$

This is Reisner's criterion; see [5, Chapter II, Section 4]. We also use the classical fact that a pure shellable complex is Cohen–Macaulay over every field [5, Chapter III, Section 2].

3.1. **The three-vertex path.** Let P have facets $\{1,2\}$ and $\{2,3\}$. It is a connected pure graph with nonempty vertex links, so (3.1) proves that it is Cohen–Macaulay over every field. Its entire face complex is

$$0 \longrightarrow S(-2)^2 \xrightarrow{\partial_2} S(-1)^3 \xrightarrow{\partial_1} S \longrightarrow 0,$$

where, in the bases (e_{12}, e_{23}) and (e_1, e_2, e_3),

$$\partial_2 = \begin{pmatrix} -x_2 & 0 \\ x_1 & -x_3 \\ 0 & x_2 \end{pmatrix}, \qquad \partial_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}. \tag{3.2}$$

The first and third rows show that ∂_2 is injective. The cycles in degree one of the full three-variable Koszul complex are generated by

$$z_{ij} = x_i e_j - x_j e_i \quad (1 \le i < j \le 3)$$

with relation $x_1 z_{23} - x_2 z_{13} + x_3 z_{12} = 0$. For completeness, this is the middle part of the exact Koszul complex for the regular sequence x_1, x_2, x_3; its exactness follows by tensoring the three two-term complexes successively, since each next variable is a non-zero-divisor modulo the previous ones. Quotienting by $\operatorname{im} \partial_2$ kills z_{12}, z_{23} and leaves the single relation $x_2 z_{13} = 0$. Consequently

$$W_1(P) = S/(x_2)(-2), \qquad W_2(P) = 0, \qquad \mathcal{R}_1(P) = V(x_2), \quad \mathcal{R}_2(P) = \varnothing. \tag{3.3}$$

The nonempty plane $V(x_2)$ and the empty next support give a negative answer to the support question of [1].

The jump loci can be computed from the same matrix. At $a \neq 0$, the transpose of (3.2) has rank two if $a_2 \neq 0$, and rank one if $a_2 = 0$. The incoming cochain map from A^0 has rank one. Therefore $\dim H^1 = 3 - 1 - \operatorname{rank} \partial_2(a)$ and $\dim H^2 = 2 - \operatorname{rank} \partial_2(a)$: both vanish in the first case and have dimension one in the second. At $a = 0$ their dimensions are three and two. Therefore

$$\mathcal{R}^1(P) = \mathcal{R}^2(P) = V(x_2), \tag{3.4}$$

although the support loci in (3.3) do not propagate.

3.2. **Nonzero adjacent modules on a shellable flag ball.** For $r \geq 4$, let D_r have vertices $t, v_1, \ldots, v_r, x$ and facets

$$\{t, v_i, v_{i+1}\} \quad (1 \leq i \leq r), \qquad \{v_1, v_2, x\}, \tag{3.5}$$

where $v_{r+1} = v_1$. It is the cone disk on an r-cycle with one triangle attached along the boundary edge v_1v_2. It is therefore a triangulation of a two-dimensional ball. The cone triangles in cyclic order, followed by the last triangle, give a shelling: each new triangle meets the preceding union in one edge, except that the last cone triangle meets it in its two radial edges. These intersections are pure of dimension one. The complex is flag, since the cycle has no triangular clique, and x is adjacent only to v_1, v_2.

Cohen–Macaulayness also follows directly from (3.1). The ball is contractible; the link of t is a cycle; the other vertex links are paths; and every edge is contained in a triangle. These are all the required lower-homology vanishing conditions.

Proposition 3.1. *For the complex (3.5), set $S = \Bbbk[t, v_1, \ldots, v_r, x]$. Then*

$$W_2(D_r) \cong S/(t)(-\mathbf{1}_{\{v_1,\ldots,v_r\}}), \qquad W_i(D_r) = 0 \quad (i \geq 3). \tag{3.6}$$

There is a surjective S-module homomorphism

$$W_1(D_r) \longrightarrow S/(v_1, \ldots, v_r)(-\mathbf{1}_{\{t,x\}}). \tag{3.7}$$

In particular both W_1, W_2 are nonzero and $\mathcal{R}_1(D_r) \not\subseteq \mathcal{R}_2(D_r)$.

Proof. Consider an induced subcomplex $(D_r)_U$. If $t \in U$, its part on $\{t, v_1, \ldots, v_r\}$ is a cone, and the part involving x, when present, is a simplex on x and those of v_1, v_2 which lie in U. The intersection is a simplex or empty. The reduced Mayer–Vietoris sequence gives zero homology in degrees one and two.

If $t \notin U$, the induced subcomplex is a subgraph of the cycle, possibly with the filled triangle on v_1, v_2, x or some of its faces. When both v_1, v_2 occur, this triangle collapses onto its base edge; when only one occurs, it contributes a pendant edge; and when neither occurs, x is isolated. Thus its first reduced homology is nonzero exactly when $\{v_1, \ldots, v_r\} \subseteq U$, and is then one-dimensional. Its second reduced homology always vanishes.

Orient the cycle by $v_i \to v_{i+1}$, with indices read modulo r. Use these orientations for the symbols $e_{v_iv_{i+1}}$, changing their signs when they disagree with the fixed vertex order. The homogeneous chain

$$\zeta = \sum_{i=1}^{r} \left(\prod_{j \notin \{i,i+1\}} v_j \right) e_{v_iv_{i+1}}$$

has multidegree $\mathbf{1}_{\{v_1,\ldots,v_r\}}$. Its boundary is zero: at e_{v_i} the contributions from the two incident oriented edges cancel. Moreover,

$$t\zeta = \partial \left(\sum_{i=1}^{r} \left(\prod_{j \notin \{i,i+1\}} v_j \right) e_{tv_iv_{i+1}} \right),$$

where the triangles are oriented in the displayed order; their radial-edge terms cancel in pairs. Thus $[\zeta]$ defines a map $S/(t)(-\mathbf{1}_{\{v_1,\ldots,v_r\}}) \to W_2(D_r)$. In each multidegree supported on all cycle vertices and excluding t, its image is the ordinary oriented cycle generator under Lemma 2.1. Adjoining x

preserves this generator, and adjoining t kills it. The preceding induced-subcomplex calculation shows that the map is an isomorphism in every multidegree. It also gives $W_i(D_r) = 0$ for $i \geq 3$.

To obtain (3.7), put $S_0 = S/(v_1, \dots, v_r) = \Bbbk[t, x]$ and regard the face complex of the two isolated vertices t, x over S_0 as an S-complex. Define an S-linear map by reducing each coefficient to S_0 and sending e_F to itself when $F \subseteq \{t, x\}$, and to zero otherwise. It is a chain map: if a discarded face loses its only discarded vertex, the corresponding boundary coefficient is that vertex variable and is zero in S_0. The cycle

$$te_x - xe_t$$

maps to the generator of the first homology of the two-point complex: the kernel of $S_0e_t \oplus S_0e_x \to S_0$, with row (t, x), is the free module generated by this cycle. Its multiples therefore prove surjectivity on homology. The target has support the coordinate plane on $\{t, x\}$. Localization of the surjection shows that this plane lies in $\mathcal{R}_1(D_r)$. By (3.6), $\mathcal{R}_2(D_r) = V(t)$, and the point with $t = 1$ and all other coordinates zero belongs to the plane and lies outside $V(t)$. □

There is also a local description of this defect. Let C be the cone subcomplex and put $a = v_1$, $b = v_2$. The relative face complex $C_\bullet(D_r)/C_\bullet(C)$ has generators $e_x, e_{ax}, e_{bx}, e_{abx}$ and is the two-variable Koszul complex on a, b, shifted by one. Since a, b form a regular sequence, its only homology is $S/(a, b)$ in degree one, with generator degree $\mathbf{1}_x$. On $C_\bullet(C)[t^{-1}]$, exterior multiplication by $t^{-1}e_t$ is a contracting homotopy: the Koszul boundary identity $\partial(e_t \wedge u) = tu - e_t \wedge \partial u$ gives $\partial h + h\partial = 1$. The relative long exact sequence gives

$$W_1(D_r)[t^{-1}] \cong S[t^{-1}]/(a, b)(-\mathbf{1}_x), \qquad W_2(D_r)[t^{-1}] = 0. \tag{3.8}$$

Thus the failure occurs on a nonempty principal open set while W_2 remains globally nonzero.

3.3. **Failure in every homological index.** Let Γ be the boundary of an r-simplex, $r \geq 1$, and let $B_r = c * \Gamma$. Write $S_\Gamma = \Bbbk[x_v : v \in V(\Gamma)]$ and $S_{B_r} = S_\Gamma[x_c]$. This is a Cohen–Macaulay r-ball. Indeed links in Γ are boundaries of simplices, and the cone links are either such boundaries or cones on them, so (3.1) applies. The face complex of Γ is the full $(r+1)$-variable Koszul complex with its last term removed. Its homology is therefore

$$W_r(\Gamma) = S_\Gamma(-\mathbf{1}_{V(\Gamma)}), \qquad W_i(\Gamma) = 0 \quad (1 \leq i < r).$$

Lemma 2.2 yields

$$W_r(B_r) = S_{B_r}/(x_c)(-\mathbf{1}_{V(\Gamma)}), \qquad W_{r+1}(B_r) = 0.$$

Thus support propagation can fail at every positive homological index. Here the next module vanishes. Proposition 3.1 gives a failure with both adjacent modules nonzero in degrees one and two.

4. Design parameters and global orientability

For a complex on $[n]$, its Alexander dual is

$$\Delta^* = \{F \subseteq [n] : [n] \setminus F \notin \Delta\}.$$

It is *bi-Cohen–Macaulay* over $\Bbbk$ if both Δ and Δ^* are Cohen–Macaulay over $\Bbbk$. Let $\dim \Delta = d - 1$, and let c be the largest integer such that every c-element vertex subset is a face. Its frame dimension is $c - 1$. For $A \subseteq [n]$, write Δ_{-A} for the induced deletion.

Definition 4.1 (Fløystad)**.** The complex Δ is an (a, b)-design if, for every pair of disjoint sets $A, B \subseteq [n]$ with $|A| = a$, $|B| = b$, the complex

$$(\mathrm{lk}_\Delta B)_{-A}$$

is bi-Cohen–Macaulay, with dimension $d - 1 - b$ and frame dimension $c - 1 - b$. All Cohen–Macaulay conditions are taken over the fixed field $\Bbbk$.

This is the definition in [3, Section 4.3], which imposes the Cohen–Macaulay conditions on the deleted links. The positive-genus closed surfaces below satisfy these local conditions.

Lemma 4.2. *A path on $m \geq 4$ vertices is bi-Cohen–Macaulay over every field. Its dimension is one and its frame dimension is zero.*

Proof. The path is a connected pure graph with nonempty vertex links, so it satisfies (3.1). Number its vertices consecutively. Its minimal nonfaces are exactly the pairs $\{i, j\}$ with $j \geq i + 2$: every larger nonface contains such a pair. The facets of its Alexander dual are therefore

$$F_{ij} = [m] \setminus \{i, j\}, \qquad i < j, \quad j \geq i + 2.$$

All have the same cardinality. Order them lexicographically by (i, j). For an earlier F_{kl} and a later F_{ij}, if $k = i$ then their intersection is already a codimension-one face of F_{ij}. If $k < i$, the facet F_{kj} is legitimate and earlier, and

$$F_{kl} \cap F_{ij} \subseteq F_{kj} \cap F_{ij} = F_{ij} \setminus \{k\}.$$

In the second case $j - k \geq 3$, so F_{kj} is indeed a facet. Every intersection with the preceding union is consequently a union of codimension-one faces. For every facet after F_{13} this union is nonempty: use $F_{1,j-1}$ when $i = 1$ and $j > 3$, and F_{1j} when $i > 1$. This proves the pure shelling criterion. The dual is pure shellable and hence Cohen–Macaulay over every field. The path is not a complete graph, so its frame dimension is zero. □

Lemma 4.3. *Let Δ be a simplicial triangulation of a connected closed surface on $n \geq 6$ vertices, with one-skeleton K_n. Then it is a $(1, 1)$-design over every field, with $d = 3$ and $c = 2$.*

Proof. For each vertex v, its link is a simplicial circle. Every other vertex occurs in that circle because the one-skeleton is complete; hence it is a cycle on $n - 1$ vertices. Deleting any $w \neq v$ leaves a path on $n - 2 \geq 4$ vertices. Lemma 4.2 proves exactly the bi-Cohen–Macaulay assertion in Definition 4.1. The original dimension is two. Every pair is an edge, but not every triple is a face: a fixed edge belongs to exactly two triangles, whereas there are $n - 2 > 2$ possible third vertices. Thus the original frame dimension is one, and the deleted link has dimension one and frame dimension zero, as required. □

Proposition 4.4. *For each $n = 12s$, $s \geq 1$, there exist two simplicial complexes Δ_+, Δ_- that are $(1, 1)$-designs over every field, with the same parameters $(n, 3, 2, 1, 1)$ and the same complete f-vector. Over a field of characteristic different from two, writing $h = (n - 3)(n - 4)/12$, their reduced homology dimensions are*

$$(\widetilde{\beta}_1, \widetilde{\beta}_2)(\Delta_+) = (2h, 1), \qquad (\widetilde{\beta}_1, \widetilde{\beta}_2)(\Delta_-) = (2h - 1, 0). \tag{4.1}$$

Proof. We use the classical triangular-embedding existence theorem for complete graphs, in the following precise range: if $n \equiv 0 \pmod{12}$, $n \geq 12$, then K_n has triangular embeddings both in the orientable surface of genus $(n - 3)(n - 4)/12$ and in the nonorientable surface of genus $(n - 3)(n - 4)/6$. This is a consequence of the complete triangular embedding theorem summarized in [2, Section 2]; its classical source is [4]. For $n = 12$, existence in both surface types is also explicit in [2, Theorem 4].

These embeddings give genuine simplicial surface triangulations. Each triangular face has three distinct vertices. Two faces cannot have the same three vertices: in a vertex link they would give the same edge twice, contrary to the link being a single cycle on $n - 1 \geq 5$ distinct neighbors. Indeed each edge incident with a vertex occurs once in its cyclic rotation, and K_n has exactly one edge from that vertex to each neighbor. Let these triangulations be Δ_+ and Δ_-. Lemma 4.3 makes both $(1, 1)$-designs over every field.

Each edge belongs to exactly two triangular faces, so both have

$$f_0 = n, \qquad f_1 = \binom{n}{2}, \qquad f_2 = \frac{2}{3}\binom{n}{2}, \qquad \chi = n - \frac{1}{3}\binom{n}{2} = 2 - 2h. \tag{4.2}$$

Their face vectors are consequently identical. The standard one-vertex cellular presentation of an orientable genus-h surface has $2h$ one-cells and one two-cell attached by a product of commutators. Its cellular two-boundary is zero, giving $H_1 = \Bbbk^{2h}$ and $H_2 = \Bbbk$. A nonorientable genus-$2h$ surface has $2h$ one-cells and a two-cell attached by the product of their squares. Its cellular two-boundary is the column $(2, \dots, 2)^{\mathsf{T}}$, of rank one when $\operatorname{char} \Bbbk \neq 2$. It gives $H_1 = \Bbbk^{2h-1}$ and $H_2 = 0$. Both surfaces are connected, so their reduced degree-zero homology vanishes; their homology in degrees above two also vanishes. This proves (4.1). □

For $n = 12$ one has $h = 6$ and (4.2) gives $(f_0, f_1, f_2) = (12, 66, 44)$. Thus the single fixed parameter set $(12, 3, 2, 1, 1)$ already disproves parameter determinacy over $\mathbb{Q}$. By (2.2), the same examples give

$$(\operatorname{rank} W_2, \operatorname{rank} W_3)(\Delta_+) = (12, 1), \qquad (\operatorname{rank} W_2, \operatorname{rank} W_3)(\Delta_-) = (11, 0).$$

Thus the generic ranks differ over each fixed field of characteristic different from two.

Remark 4.5. In characteristic two the cellular two-boundary of the nonorientable surface is zero. Both surfaces then have Betti pair $(2h, 1)$. The Betti-number distinction in Proposition 4.4 therefore holds in characteristic different from two. The local design conditions hold in every characteristic.

5. Support, generic rank and local design conditions

Proposition 3.1 answers the support-propagation question of [1, Section 5], even if both modules under comparison are required to be nonzero. In (3.6) and (3.7), the induced cycle generates W_2, and the missing edge gives a quotient of W_1 supported on a coordinate plane extending outside $V(t)$. These induced-subcomplex homology classes occur in a shellable Cohen–Macaulay ball. The path calculation (3.4) also exhibits equal jump loci alongside unequal support loci.

Proposition 4.4 answers the parameter-determinacy question in [3, Problem 4]. The two surface types have the same deleted links and face numbers; their cellular two-boundaries have different ranks because their orientability differs. Among the connected neighborly closed surfaces considered here, n and orientability determine the Betti numbers in characteristic different from two. We have not yet classified the homology of arbitrary (a, b)-designs.

Both answers follow from the face complex (1.1). Its strands recover induced-subcomplex homology, and its generic rank recovers global homology. The counterexamples show that local Cohen–Macaulay conditions and numerical design parameters leave these invariants undetermined.

References

[1] M. Aprodu, G. Farkas, C. Raicu, A. Sammartano, and A. I. Suciu, *Higher resonance schemes and Koszul modules of simplicial complexes*, J. Algebraic Combin. **59** (2024), 787–805. arXiv:2309.00609.

[2] M. N. Ellingham and C. Stephens, *Triangular embeddings of complete graphs (neighborly maps) with 12 and 13 vertices*, J. Combin. Des. **13** (2005), 336–344. doi:10.1002/jcd.20048.

[3] G. Fløystad, *Enriched homology and cohomology modules of simplicial complexes*, J. Algebraic Combin. **25** (2007), 285–307. doi:10.1007/s10801-006-0038-z; arXiv:math/0411570.

[4] G. Ringel, *Map color theorem*, Grundlehren der mathematischen Wissenschaften 209, Springer-Verlag, Berlin–Heidelberg–New York, 1974.

[5] R. P. Stanley, *Combinatorics and commutative algebra*, second edition, Progress in Mathematics 41, Birkhäuser, Boston, 1996.