EXTERIOR ANNIHILATORS BEYOND ODD-FACTOR EXPLANATIONS

A COUNTEREXAMPLE TO QUESTION 6.2, AN EXPLICIT (9, 3) WITNESS FOR QUESTION 6.3, AND NONUNIMODAL ANNIHILATOR HOMOLOGY

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Abstract. We give explicit negative answers to two structural possibilities for special forms in an exterior algebra. In nine variables, a cubic obtained by packaging the primitive kernel of a symplectic quadratic form has a quadratic annihilator whose square is nonzero. This annihilator cannot be explained by a product of odd-degree generators, giving a counterexample to Question 6.2 of Lundqvist–Nicklasson. The same cubic belongs to a linear-generated ideal precisely at a minimal generator-space dimension of three; it is an explicit witness for their Question 6.3 at $(n, d) = (9, 3)$. Next, for every $m \geq 24$, the cubic $x \sum_{j=1}^{m} a_j b_j$ in $2m + 1$ variables has annihilator homology $\mathrm{Ann}(f)/(f)$ whose symmetric Hilbert function decreases before its midpoint. A tensor construction removes all linear factors in $2m + 4$ variables for $m \geq 31$. A further construction combines nonunimodality with an annihilator that still admits no odd-factor explanation. These are counterexamples for special forms; they make no assertion against a conjectural Hilbert function of general forms or against the coefficientwise lower-bound question.

1. Questions about odd forms and the statements proved here

All exterior algebras in this note are over $\mathbb{C}$. For a vector space V of dimension n, let $E = \bigwedge V$ with its ordinary degree grading; multiplication will usually be written without a wedge. For a homogeneous form f, define

$$\mathrm{Ann}(f) = \{g \in E : gf = 0\}.$$

If f has positive odd degree d, graded commutativity gives $f^2 = 0$. Thus $(f) \subseteq \mathrm{Ann}(f)$, and the graded vector space

$$\mathsf{H}(f) = \mathrm{Ann}(f)/(f) \tag{1.1}$$

is the homology of the multiplication differential $d_f = f \wedge - : E_i \to E_{i+d}$. Its Hilbert function is $h_i(f) = \dim \mathsf{H}(f)_i$.

Two questions in [1, Section 6] concern the explanation of annihilators by odd-degree forms. Their Question 6.2 asks whether

$$gf = 0, \qquad \deg g < \frac{n - \deg f}{2} \tag{1.2}$$

forces the existence of positive odd-degree forms $h_1, \ldots, h_s$ with

$$f \in (h_1, \ldots, h_s), \qquad g \in (h_1 \cdots h_s). \tag{1.3}$$

Question 6.3 asks whether there are forms f outside every such ideal when the generator degrees are also less than $\deg f$ and $\deg(h_1 \cdots h_s) < (n - \deg f)/2$. Whenever the degree of a product is used, the product is assumed nonzero. For the first question this is automatic if $g \neq 0$ and $g \in (h_1 \cdots h_s)$.

After Question 6.5, the same paper asks an unnumbered auxiliary question: is the symmetric Hilbert function of (1.1) weakly increasing up to degree $n/2$? The theorems below disprove that

Date: Research note / October 2026.

2020 *Mathematics Subject Classification.* Primary 13A02; Secondary 13D02, 15A75.

Key words and phrases. exterior algebra, annihilator, odd form, homology, Hilbert function, unimodality.

proposed monotonicity. The numbered Question 6.5 concerns a coefficientwise lower bound for $E/(f)$, a separate assertion with different quantifiers.

Theorem 1.1 (A primitive-kernel cubic)**.** *Let $V = W \oplus Z$, where*

$$W = \langle u_1, u_2, u_3, u_4 \rangle, \qquad Z = \langle z_1, z_2, z_3, z_4, z_5 \rangle,$$

and put

$$\begin{aligned} g &= u_1u_2 + u_3u_4, \\ \alpha_1 &= u_1u_2 - u_3u_4, & \alpha_2 &= u_1u_3, & \alpha_3 &= u_1u_4, \\ \alpha_4 &= u_2u_3, & \alpha_5 &= u_2u_4, \end{aligned}$$

$$f_0 = \sum_{j=1}^{5} \alpha_j z_j. \tag{1.4}$$

Then $gf_0 = 0$ and $g^2 \neq 0$, and the pair (f_0, g) admits no explanation of the form (1.3). Moreover,

$$\min\{\dim U : U \subseteq V,\ f_0 \in (U)\} = 3. \tag{1.5}$$

Consequently Question 6.2 has a negative answer, and f_0 is a witness for Question 6.3 at $(n, d) = (9, 3)$.

Theorem 1.2 (Nonunimodal annihilator homology)**.** *Let $W_m = \langle a_1, b_1, \dots, a_m, b_m \rangle$ and $\omega_m = \sum_{j=1}^{m} a_j b_j$.*

(1) *For $m \geq 24$, the cubic $f = x\omega_m$ in $2m + 1$ variables satisfies*

$$h_{m-2}(f) > h_{m-1}(f), \qquad m - 1 < (2m + 1)/2.$$

For $m = 24$ these coefficients are 10772391370048 *and* 10649977831752*.*

(2) *For $m \geq 31$, the cubic $F = abc + x\omega_m$ on disjoint variable spaces, of total dimension $2m + 4$, has no linear factor and satisfies*

$$h_{m-1}(F) > h_m(F), \qquad m < (2m + 4)/2.$$

For $m = 31$ these coefficients are 816399943529769648 *and* 810144005265173712*.*

Theorem 1.3 (Nonunimodality with an unexplained annihilator)**.** *For every $m \geq 144$, put $F = f_0 + x\omega_m$ on disjoint spaces, so that $\dim V = 2m + 10$, and set $G = xg$ with f_0, g as in Theorem 1.1. Then F has no linear factor,*

$$h_m(F) > h_{m+1}(F), \qquad m + 1 < (2m + 10)/2,$$

and $GF = 0$. No collection of positive odd-degree forms satisfies

$$F \in (h_1, \dots, h_s), \qquad G \in (h_1 \cdots h_s).$$

The cubic annihilator G obeys $\deg G < (\dim V - \deg F)/2$.

The mechanism for the first theorem retains all directions of a primitive exterior-power kernel. The mechanism for the second is different: a closed formula for a symplectic multiplication map gives an explicit decline in a difference of binomial coefficients. The third theorem joins these constructions by a graded tensor product. Every claim about the displayed forms is proved below without a computer calculation.

2. A quadratic annihilator with nonzero square

Keep the spaces W, Z of Theorem 1.1. The multiplication map

$$g \wedge - : \bigwedge^2 W \longrightarrow \bigwedge^4 W$$

is nonzero, so its kernel

$$P = \ker(g \wedge -) \tag{2.1}$$

has dimension five. The forms $\alpha_1, \dots, \alpha_5$ are linearly independent and lie in P. Indeed,

$$(u_1u_2 + u_3u_4)(u_1u_2 - u_3u_4) = 0,$$

while every term of $g\alpha_j$ for $j \geq 2$ repeats a variable. Thus these forms are a basis of P, and

$$gf_0 = 0, \qquad g^2 = 2u_1u_2u_3u_4 \neq 0. \tag{2.2}$$

Proposition 2.1. *The pair (f_0, g) has no odd-factor explanation (1.3).*

Proof. Suppose $g \in (h_1 \cdots h_s)$ with positive odd homogeneous degrees. Since $g \neq 0$ and $\deg g = 2$, their nonzero product has degree at most two. Hence every h_i is linear and s is either one or two. If $s = 1$, the degree-two part of the ideal is h_1V, so $g = h_1v$ for a linear v. If $s = 2$, then $g = ch_1h_2$ for a scalar c. In either case g is decomposable and $g^2 = 0$, contradicting (2.2). The condition on f_0 is not even needed for this contradiction. □

Here $\deg g = 2 < (9-3)/2 = 3$, so the example lies in the strict range asked in Question 6.2.

Lemma 2.2 (Linear factors and linear annihilators)**.** *For $0 \neq \ell \in V$ and any homogeneous $f \in \bigwedge V$, $\ell f = 0$ if and only if $f \in (\ell)$.*

Proof. Choose $V = \mathbb{C}\ell \oplus V'$ and write $f = \alpha + \ell\beta$ with $\alpha, \beta \in \bigwedge V'$. Then $\ell f = \ell\alpha$, which vanishes exactly when $\alpha = 0$, since multiplication by ℓ embeds $\bigwedge V'$ into $\ell \bigwedge V'$. □

Lemma 2.3. *The cubic f_0 has no nonzero linear annihilator and hence no linear factor.*

Proof. Write a linear form as $u + z$ with $u \in W$, $z \in Z$. The equations in distinct bidegrees for $(u+z)f_0 = 0$ are $uf_0 = 0$ and $zf_0 = 0$. Since the α_j are independent, $zf_0 = 0$ forces $z \wedge z_j = 0$ for all j, which gives $z = 0$. The equation $uf_0 = 0$ forces $u\alpha_2 = u\alpha_5 = 0$. The two kernels on W are respectively $\langle u_1, u_3\rangle$ and $\langle u_2, u_4\rangle$; their intersection is zero. Thus $u = 0$. Apply Lemma 2.2 for the factor statement. □

3. The exact linear-ideal threshold of the cubic

For a subspace $U \subseteq V$, the ideal (U) is the kernel of the quotient homomorphism $\bigwedge V \to \bigwedge(V/U)$. We therefore prove (1.5) by tracking the image of f_0 in every quotient by a two-dimensional subspace of the generator space.

Lemma 3.1. *For $T \subseteq W$ with $t = \dim T \in \{0, 1, 2\}$, let $\pi_T : P \to \bigwedge^2(W/T)$ be induced by quotient. Then*

t	0	1	2
$\dim \ker \pi_T$	0	2	4.

Proof. For $t = 0$ the map is inclusion. If $t = 1$ and $T = \mathbb{C}u$, then the kernel of $\bigwedge^2 W \to \bigwedge^2(W/T)$ is $u \wedge W$, of dimension three. Multiplication by g maps W isomorphically onto $\bigwedge^3 W$: the images of its four displayed basis vectors are nonzero scalar multiples of the four complementary exterior monomials. Thus $gu \neq 0$. The perfect pairing $\bigwedge^3 W \otimes W \to \bigwedge^4 W$ provides a v with $guv \neq 0$. The functional $g \wedge -$ is consequently nonzero on $u \wedge W$, and its kernel there has dimension two. That kernel is $P \cap (u \wedge W) = \ker \pi_T$.

For $t = 2$ the target has dimension one. If $\pi_T = 0$, the five-dimensional hyperplane P would equal the kernel of $\bigwedge^2 W \to \bigwedge^2(W/T)$. The latter hyperplane is defined by a nonzero decomposable element of $\bigwedge^2 W^*$, which has rank two as a skew form. By contrast, after identifying $\bigwedge^4 W$ with $\mathbb{C}$ using $u_1u_2u_3u_4$, the functional $\alpha \mapsto g\alpha$ is defined by $u_1^*u_2^* + u_3^*u_4^*$, of rank four. Two defining functionals for the same hyperplane must be proportional, which contradicts these ranks. Thus π_T has rank one and kernel dimension four. □

Proposition 3.2. *The minimum in* (1.5) *is exactly three.*

Proof. Let $U \subseteq V$ have dimension two and put $T = U \cap W$, $t = \dim T$. In V/U there is a canonical subspace

$$W' = (W + U)/U \cong W/T$$

with quotient

$$Z' = V/(W + U) \cong Z/\operatorname{pr}_Z(U), \qquad \dim Z' = 3 + t.$$

The image of f_0 belongs to $\bigwedge^2 W' \cdot (V/U)$. Modulo $\bigwedge^3 W'$, this has the canonical identification

$$(\bigwedge^2 W' \cdot (V/U))/\bigwedge^3 W' \cong \bigwedge^2 W' \otimes Z'.$$

Identify Z^* with P by $z_j^* \mapsto \alpha_j$. The tensor represented by f_0 in this quotient corresponds to the map

$$(Z')^* \hookrightarrow Z^* \cong P \xrightarrow{\pi_T} \bigwedge^2 (W/T). \tag{3.1}$$

This expression is independent of any chosen splitting of V/U; changes of splitting add terms in $\bigwedge^3 W'$, which have already been removed.

If $f_0 \in (U)$, then its image is zero, so (3.1) is zero. Its first map is an injection from a space of dimension $3 + t$, whereas Lemma 3.1 gives kernel dimensions $0, 2, 4$ for $t = 0, 1, 2$. In all three cases $3 + t > \dim\ker \pi_T$, a contradiction. Thus f_0 is in no ideal generated by a two-dimensional subspace. This also excludes one-dimensional subspaces, by extending any such subspace to a two-dimensional one.

On the other hand, take any three-dimensional $U_0 \subseteq W$. Then $\dim(W/U_0) = 1$, so $\bigwedge^2(W/U_0) = 0$. Since $f_0 \in \bigwedge^2 W \cdot Z$, its image in $\bigwedge^3(V/U_0)$ is zero. Hence $f_0 \in (U_0)$ and the minimum is three. □

Proof of Theorem 1.1. Equations (2.2) and Proposition 2.1 prove the counterexample. Proposition 3.2 proves the exact threshold. For a cubic, the positive odd generator degrees less than three are all one. A nonzero product with degree less than $(9 - 3)/2 = 3$ has at most two such generators. The threshold excludes every corresponding ideal, giving the claimed Question 6.3 witness. □

Remark 3.3. Existence in Question 6.3 at $(9, 3)$ also follows by dimension count. For a fixed two-plane U, the cubic part of (U) has dimension $\binom{9}{3} - \binom{7}{3} = 49$. The incidence over $\mathrm{Gr}_2(V)$ therefore has dimension $49 + 14 = 63 < 84 = \dim \bigwedge^3 V$. A general cubic avoids all these ideals. The point of the explicit f_0 is that it does so while retaining a prescribed nondecomposable quadratic annihilator. We have not yet classified the full set of parameters in Question 6.3.

4. Symplectic exterior multiplication

Let W_m and ω_m be as in Theorem 1.2. Write

$$L_i : \bigwedge^i W_m \longrightarrow \bigwedge^{i+2} W_m, \qquad \alpha \longmapsto \omega_m \alpha.$$

Throughout, $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.

Lemma 4.1. *For every integer i,*

$$\operatorname{rank} L_i = \min\left\{\binom{2m}{i}, \binom{2m}{i+2}\right\}. \tag{4.1}$$

Proof. Make the exterior monomial basis orthonormal for a Hermitian inner product. Let L be multiplication by ω_m and let Λ be its adjoint, the sum of contractions that remove the pairs $a_j b_j$. On the four-dimensional algebra $\bigwedge\langle a_j, b_j\rangle$, the commutator $[\Lambda_j, L_j]$ acts by $1, 0, 0, -1$ on $1, a_j, b_j, a_j b_j$. Summing over the m commuting even operators gives

$$[\Lambda, L]\big|_{\bigwedge^i W_m} = (m-i)\,\mathrm{id}\,. \tag{4.2}$$

For $i < m$ and $\alpha \in \bigwedge^i W_m$, this implies

$$\|L\alpha\|^2 = \|\Lambda\alpha\|^2 + (m-i)\|\alpha\|^2.$$

Hence L_i is injective for $i < m$. Exterior pairing into $\bigwedge^{2m} W_m$ identifies the transpose of L_i, up to a nonzero sign, with L_{2m-i-2}. For $i \geq m-1$, this latter source degree is at most $m-1$, so its map is injective and L_i is surjective. At $i = m-1$ both conditions hold and the map is an isomorphism. These ranks are exactly (4.1). □

5. The homology of a symplectic cubic and its decline

Take $V = \mathbb{C}x \oplus W_m$ and $f = x\omega_m$. Under $\bigwedge V = \bigwedge W_m \oplus x \bigwedge W_m$,

$$d_f(\alpha) = x\omega_m\alpha, \qquad d_f(x\beta) = 0.$$

Thus in every degree i,

$$\mathsf{H}(f)_i \cong \ker L_i \;\oplus\; x\left(\bigwedge^{i-1} W_m/\omega_m \bigwedge^{i-3} W_m\right). \tag{5.1}$$

In particular, if $r_i = \operatorname{rank} L_i$, then

$$h_i(f) = \binom{2m+1}{i} - r_i - r_{i-3}. \tag{5.2}$$

Lemma 4.1 supplies all coefficients, including the middle ones. When $0 \leq i \leq m-1$, both relevant maps are injective, so

$$h_i(f) = \binom{2m}{i-1} - \binom{2m}{i-3}. \tag{5.3}$$

Lemma 5.1. *For every odd-degree $f \in \bigwedge^d V$, $h_i(f) = h_{n-i}(f)$.*

Proof. Write $r_i(f)$ for the rank of $f \wedge - : E_i \to E_{i+d}$. Perfect pairing into E_n makes its transpose, up to sign, multiplication in degree $n-i-d$. Thus $r_i(f) = r_{n-i-d}(f)$. Since

$$h_i(f) = \binom{n}{i} - r_i(f) - r_{i-d}(f),$$

substitution gives the symmetry. □

Put $P_j = \binom{2m}{j} - \binom{2m}{j-2}$. For $0 \leq j \leq m-1$, direct cancellation of factorials gives

$$\frac{P_{j+1}}{P_j} = \frac{(2m-2j)(2m+2-j)}{(j+1)(2m+2-2j)}. \tag{5.4}$$

One way to see the cancellation is to write

$$P_j = \binom{2m}{j}\frac{(2m+1)(2m+2-2j)}{(2m-j+2)(2m-j+1)}.$$

All denominators and all P_j in the stated range are positive.

Proof of Theorem 1.2(1). Equation (5.3) says $h_i(f) = P_{i-1}$ for $i \le m-1$. Substitute $j = m-3$ into (5.4) to get

$$\frac{h_{m-1}(f)}{h_{m-2}(f)} = \frac{3(m+5)}{4(m-2)}.$$

This is less than one exactly when $m > 23$. Both indices precede $(2m+1)/2$. At $m = 24$ formula (5.3) gives

$$h_{22}(f) = \binom{48}{21} - \binom{48}{19} = 10772391370048,$$

$$h_{23}(f) = \binom{48}{22} - \binom{48}{20} = 10649977831752.$$

Their difference is 122413538296. Together with Lemma 5.1, the strict decline before the midpoint proves nonunimodality. □

Remark 5.2. The number 49 is the first variable count where these particular adjacent degrees decline in this family: at $m = 23$ they are equal. We have not yet determined the smallest variable count over all exterior forms.

Proposition 5.3 (Failure of three-step monotonicity)**.** *For $m \ge 58$, the same cubic has $h_{m-2}(f) < h_{m-5}(f)$, with $m - 2 < (n-3)/2$.*

Proof. Multiply three consecutive ratios in (5.4). This gives

$$\frac{h_{m-2}(f)}{h_{m-5}(f)} = \frac{4(m+8)(m+7)(m+6)}{7(m-5)(m-4)(m-3)}.$$

It is less than one if and only if $m^3 - 56m^2 - 85m - 588 > 0$. At $m = 58 + t$, this polynomial is $t^3 + 118t^2 + 3511t + 1210$, positive for $t \ge 0$. Here $n = 2m+1$, so $(n-3)/2 = m - 1 > m - 2$. □

6. A graded tensor formula and the removal of linear factors

Lemma 6.1 (Odd multiplication Künneth formula)**.** *Let V_1, V_2 be disjoint vector spaces and $f_j \in \bigwedge^3 V_j$. For $F = f_1 + f_2$,*

$$\mathsf{H}(F) \cong \mathsf{H}(f_1) \otimes \mathsf{H}(f_2) \tag{6.1}$$

as graded vector spaces, where the tensor product has total exterior degree. In particular, $\mathrm{Hilb}_{\mathsf{H}(F)}(t) = \mathrm{Hilb}_{\mathsf{H}(f_1)}(t)\,\mathrm{Hilb}_{\mathsf{H}(f_2)}(t)$.

Proof. The exterior tensor identification puts the differential in the form

$$d_F(a \otimes b) = d_{f_1} a \otimes b + (-1)^{\deg a} a \otimes d_{f_2} b.$$

Each differential raises degree by three and is odd. In any finite-dimensional graded complex with differential of degree three, choose homogeneous complements to boundaries inside cycles, and to cycles inside the whole space. It decomposes into the homology with zero differential and two-term summands $u \mapsto v$, with $\deg v = \deg u + 3$. On a two-term summand, define the contraction $s(v) = u$, $s(u) = 0$. It has odd degree -3 and satisfies $ds + sd = \mathrm{id}$.

When such a summand is tensored with the other complex, $s \otimes 1$ still contracts the tensor differential: its odd degree makes the two mixed terms cancel. Thus all summands having a two-term factor are contractible. Only the tensor product of the two zero-differential homology summands survives, with its original total grading. This proves (6.1). □

Lemma 6.2. *If $0 \ne f_j \in \bigwedge^3 V_j$ on disjoint spaces, then $f_1 + f_2$ has no nonzero linear annihilator.*

Proof. Write a linear form as $u+v$ with $u \in V_1$, $v \in V_2$. The four products uf_1, vf_1, uf_2, vf_2 have respective bidegrees $(4,0),(3,1),(1,3),(0,4)$. If their sum is zero, each bidegree vanishes. In particular $vf_1 = 0$ and $uf_2 = 0$. Exterior multiplication between disjoint spaces is tensor multiplication, so the nonzero f_j force $v = 0$ and $u = 0$. □

Proof of Theorem 1.2(2). For $f_1 = abc$ on the three-dimensional space $\langle a, b, c\rangle$, multiplication has only one nonzero map, $E_0 \to E_3$, and that is an isomorphism. Thus

$$\operatorname{Hilb}_{\mathsf{H}(abc)}(t) = 3t + 3t^2.$$

Put $f_2 = x\omega_m$ and write $h_i = h_i(f_2)$, $q_i = h_i(F)$. Lemma 6.1 gives $q_i = 3(h_{i-1} + h_{i-2})$. Therefore

$$\begin{aligned} q_{m-1} - q_m &= 3(h_{m-3} - h_{m-1}) \\ &= 3\left(2\binom{2m}{m-4} - \binom{2m}{m-6} - \binom{2m}{m-2}\right) \\ &= 3\binom{2m}{m-2}\frac{4(m-30)(m+1)(2m+1)}{(m+3)(m+4)(m+5)(m+6)}. \end{aligned} \tag{6.2}$$

To verify the last factorization, divide by $\binom{2m}{m-2}$ and clear the displayed denominator. The numerator before factorization is

$$\begin{aligned} &2(m-2)(m-3)(m+5)(m+6) \\ &\quad - (m-2)(m-3)(m-4)(m-5) - (m+3)(m+4)(m+5)(m+6), \end{aligned}$$

which expands to $4(m-30)(m+1)(2m+1)$. For $m \geq 31$, expression (6.2) is positive, and $m < n/2 = m+2$. At $m = 31$, the binomial formulas give

$$q_{30} = 816399943529769648, \qquad q_{31} = 810144005265173712.$$

Finally, Lemmas 6.2 and 2.2 show that F has no linear factor. □

Since a positive odd degree less than three is necessarily one, this family has no proper positive odd-degree factor. Thus such factors are not responsible for every failure of midpoint monotonicity.

7. Keeping the primitive annihilator in a nonunimodal family

Let $c_j = h_j(f_0)$ for the nine-variable cubic in (1.4). We require only its support, not its complete Hilbert function. Because $f_0 \neq 0$ and Lemma 2.3 excludes linear annihilators, $c_0 = c_1 = 0$. Equation (2.2) supplies a nonzero class of g in degree two, since the ideal (f_0) has no degree-two part. Hence $c_2 > 0$. By Lemma 5.1, $c_j = c_{9-j}$, so

$$c_j = 0 \quad \text{unless } 2 \leq j \leq 7, \qquad c_j \geq 0, \qquad c_2 > 0. \tag{7.1}$$

Now put $F = f_0 + x\omega_m$ on disjoint variable spaces. If $H_i = h_i(x\omega_m)$, then Lemma 6.1 gives

$$h_i(F) = \sum_{j=2}^{7} c_j H_{i-j}. \tag{7.2}$$

Proposition 7.1. *For $m \geq 144$, $h_m(F) > h_{m+1}(F)$, and F has no linear factor.*

Proof. For $2 \leq j \leq 7$, both indices $m-j$ and $m+1-j$ are at most $m-1$, so (5.3) applies. Set $a = j+1$; then $3 \leq a \leq 8$, and (5.4) gives

$$\frac{H_{m+1-j}}{H_{m-j}} = \frac{P_{m-a+1}}{P_{m-a}} = \frac{a(m+a+2)}{(a+1)(m-a+1)}.$$

This is less than one exactly when $m > 2a^2 + 2a - 1$. The largest right-hand threshold for $3 \leq a \leq 8$ is 143. Thus every summand in (7.2) decreases at the comparison in question when $m \geq 144$. The

nonnegative c_j and $c_2 > 0$ make the total decline strict. Both comparison indices are less than $n/2 = m + 5$. The no-factor assertion follows from Lemmas 6.2 and 2.2. □

Lemma 7.2. *For $G = xg$ in the ambient space of F, its linear annihilator space is exactly $\mathbb{C}x$. In particular G is not a decomposable product of three linear forms.*

Proof. Write the ambient space as $\mathbb{C}x \oplus W \oplus R$, where W is the four-dimensional primitive space and R consists of all remaining variables. In the quotient by x, multiplication by g on degree one is injective. On W it is the isomorphism $W \to \bigwedge^3 W$ used in Lemma 3.1. On R it is the injection $v \mapsto g \otimes v$ into $\bigwedge^2 W \otimes R$. The two images have distinct bidegrees. Thus $\ell xg = 0$ forces the image of ℓ modulo x to be zero, and every multiple of x plainly annihilates G. A nonzero decomposable cubic has three independent linear factors, each a linear annihilator, whereas G has only one annihilator direction. □

Proposition 7.3. *The pair (F, G) has no odd-factor explanation.*

Proof. First,

$$GF = xgf_0 + xgx\omega_m = 0,$$

and $G \neq 0$. Suppose $G \in (h_1 \cdots h_s)$ for positive odd-degree homogeneous forms. The nonzero product has total degree at most three. If its degree is one, there is a single linear h_1 dividing G. Lemma 2.2 and Lemma 7.2 force h_1 to be proportional to x. But $F \notin (x)$ by Proposition 7.1.

If the product has degree two, it consists of two linear forms; G is then their product times a linear form, making G decomposable, contrary to Lemma 7.2. If the total degree is three with three linear generators, it again makes G decomposable. The remaining possibility is one cubic generator h_1. Then G is a scalar multiple of h_1; because F also has degree three and belongs to (h_1), F would be proportional to G. This is impossible: $xG = 0$ but F has no nonzero linear annihilator. These cases exhaust all positive odd degrees with total at most three. □

Proof of Theorem 1.3. Apply Propositions 7.1 and 7.3. Here $n = 2m + 10$ and

$$\deg G = 3 < \frac{n - \deg F}{2} = m + \frac{7}{2},$$

so the annihilator is within the strict range of Question 6.2. □

8. Relation to Hilbert-function bounds

The symmetric function studied here is the Hilbert function of the homology (1.1). Its relation to the Hilbert function of $E/(f)$ clarifies the distinction between the two monotonicity questions.

Proposition 8.1. *Let f have odd degree d and set $a_i = \dim(E/(f))_i$, $H_i = h_i(f)$, with all negative-index values zero. Then*

$$a_i = \binom{n}{i} - a_{i-d} + H_{i-d}. \tag{8.1}$$

Proof. Let r_i be the rank of multiplication by f in degree i. Then $a_i = \binom{n}{i} - r_{i-d}$ and $H_{i-d} = \binom{n}{i-d} - r_{i-d} - r_{i-2d}$. Since $a_{i-d} = \binom{n}{i-d} - r_{i-2d}$, we have $r_{i-d} = a_{i-d} - H_{i-d}$. Substitute this into the first equality. □

Iterating (8.1) shows that changes in a predicted quotient sequence depend on alternating sums of the earlier H_i. Proposition 5.3 compares two isolated terms; controlling the full sums requires additional estimates. Moreover, the initial index in the argument following Question 6.5 of [1] is the first nonzero homology degree; for $x\omega_m$ that index is one, whereas the declines proved above occur later.

Thus the results answer Question 6.2, provide an explicit fixed parameter witness for Question 6.3, and negate the unnumbered midpoint-monotonicity question. We have not yet resolved Question 6.1

for general forms or the coefficientwise inequality in Question 6.5. For a fixed degree, the ranks of all multiplication matrices are maximal on an open subset of the form parameter space. Consequently a special form with extra annihilators cannot by itself demonstrate an extra annihilator for a general form. The results here are structural counterexamples to assertions quantified over all forms.

References

[1] S. Lundqvist and L. Nicklasson, *On generic principal ideals in the exterior algebra*, J. Pure Appl. Algebra **223** (2019), 2615–2634, doi:10.1016/j.jpaa.2018.09.011. The question numbering and formulations used here are those of arXiv:1803.03563v1, Section 6.