FINITE QUOTIENTS AND COHOMOLOGY BEYOND GOODNESS

NEGATIVE ANSWERS TO QUESTION 5.15 OF COHEN ET AL. AND SUCIU'S QUESTION 4.2, WITH FINITE-COEFFICIENT RECOVERY IN DIMENSION TWO

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Abstract. For every $k \geq 3$, the Stallings–Bieri kernel of $(F_2)^k \to \mathbb{Z}$ is an arrangement group that fails Serre goodness in degree k. With coefficients in $\mathbb{F}_p$, its ordinary top cohomology consists of all bilateral sequences, and the profinite comparison image consists exactly of the periodic sequences. A complement of five affine lines realizes the first member of this family, giving negative answers to the arrangement-goodness questions of Cohen et al. and Suciu. We also prove that two connected finite two-dimensional CW-complexes with isomorphic profinite fundamental groups and equal Euler characteristic have naturally isomorphic cohomology functors on finite local coefficient modules. The degree-two isomorphism is noncanonical. Finite derivations establish exactness of the completed presentation matrices; stabilization and finite-ring cancellation then identify their transpose modules. Corresponding finite covers have isomorphic integral homology. For complex line-arrangement complements, the surjective rational cup product in degree two makes the Euler characteristic itself an invariant of the profinite fundamental group.

1. Arrangement goodness and finite-coefficient recovery

For a discrete group G, write $\alpha_G : G \to \widehat{G}$ for its profinite completion. A finite G-module means an abelian group M that is finite as a set, with a left G-action. The action factors through a finite group and therefore extends uniquely to a continuous $\widehat{G}$-action. Continuous cohomology is computed with continuous inhomogeneous cochains on the profinite group and the discrete module.

Definition 1.1. A group G is *good*, in the sense of Serre [8], if the restriction maps

$$\alpha_G^* : H^q_{\mathrm{cont}}(\widehat{G}; M) \longrightarrow H^q(G; M) \tag{1.1}$$

are isomorphisms for every finite G-module M and every $q \geq 0$.

Question 5.15, attributed to Suciu in the draft monograph of Cohen, Denham, Falk, Schenck, Suciu, Terao and Yuzvinsky [2, p. 99], asks: "Are arrangement groups good?" The same unrestricted question appears as Question 4.2 in [9]. An arrangement group is the fundamental group of the complement of a finite complex hyperplane arrangement. Both questions range over all such groups, including those with nonaspherical complements. For a nonaspherical complement X, the group cohomology $H^*(\pi_1 X; M)$ and the space cohomology $H^*(X; M)$ can differ.

For $k \geq 3$, put

$$A_k = \prod_{i=1}^{k} F(a_i, b_i), \qquad \chi(a_i) = \chi(b_i) = 1, \qquad S_k = \ker(\chi : A_k \to \mathbb{Z}). \tag{1.2}$$

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These are the classical Stallings–Bieri kernels. Their arrangement realizations are due to Artal Bartolo, Cogolludo-Agustín and Matei [1]. Their geometric realization, together with the cohomology calculation below, gives the required counterexamples.

Theorem 1.2 (Negative answer to the arrangement-goodness question)**.** *For every $k \geq 3$ and every prime p,*

$$H^k(S_k; \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[t, t^{-1}], \mathbb{F}_p) \cong \mathbb{F}_p^{\mathbb{Z}}.$$

Under this identification, the image of

$$H^k_{\mathrm{cont}}(\widehat{S}_k; \mathbb{F}_p) \longrightarrow H^k(S_k; \mathbb{F}_p)$$

is exactly the subgroup of periodic bilateral sequences. In particular, S_k is not good. Each S_k is the group of a projective hyperplane arrangement, and also of an affine hyperplane arrangement. The group S_3 is realized by the complement of the five affine lines

$$y = 0, \quad z = 0, \quad y = 2, \quad y = z, \quad z = 1 \tag{1.3}$$

in $\mathbb{C}^2$. Central hyperplane arrangements with group $S_k \times \mathbb{Z}$ are not good either.

Throughout, a *finite two-dimensional complex* is a connected finite CW-complex with cells only in dimensions zero, one and two. The results also apply to spaces homotopy equivalent to such complexes.

For $G_X = \pi_1(X)$ and $G_Y = \pi_1(Y)$, fix an isomorphism

$$\phi : \widehat{G}_X \xrightarrow{\sim} \widehat{G}_Y. \tag{1.4}$$

If M is a finite G_Y-module with extended action $\widehat{\rho}$, its transport M^ϕ has G_X-action $\widehat{\rho} \circ \phi \circ \alpha_{G_X}$. A homomorphism of finite G_Y-modules is transported by the same underlying homomorphism. Equivalently, after identifying the two completions with P, both spaces have cohomology functors on the category $\mathsf{Fin}(P)$ of finite continuous left P-modules.

Theorem 1.3 (Recovery in dimension two)**.** *Let X, Y be finite two-dimensional complexes with $\chi(X) = \chi(Y)$ and an isomorphism (1.4). There exist isomorphisms*

$$H^q(X; M^\phi) \cong H^q(Y; M), \qquad q \geq 0, \tag{1.5}$$

simultaneously natural in every finite module M. In degrees zero and one the isomorphisms are canonical for the specified ϕ. The degree-two isomorphism is noncanonical. The groups vanish in higher degrees.

For corresponding connected finite covers X_U, Y_U determined by open subgroups of the identified completion,

$$H_q(X_U; \mathbb{Z}) \cong H_q(Y_U; \mathbb{Z}), \qquad q \geq 0. \tag{1.6}$$

These integral isomorphisms are assertions about abstract groups.

Naturality in (1.5) means compatibility with every homomorphism of finite coefficient modules. In degree two the transformation is obtained by choosing one completed-module isomorphism and tensoring it with all coefficients. Its dependence on that choice is discussed in the final section.

Corollary 1.4 (Line arrangements)**.** *Let X, Y be complements of complex affine line arrangements, or of nonempty projective line arrangements. If their profinite fundamental groups are isomorphic, then their Euler characteristics are equal and all conclusions of Theorem 1.3 hold.*

The Euler calculation used in Theorem 1.3 is related to the description of local-system homology for sufficiently generic sections of aspherical arrangements in terms of the fundamental group and Euler characteristic [4]. To obtain a simultaneous natural comparison for all finite modules, we use completed presentation matrices and cancellation over their finite quotient rings.

2. The Laurent orbit in a Stallings kernel

The following countability obstruction applies to every finitely generated group.

Lemma 2.1. *For a finitely generated group G, a finite G-module M and $q \geq 0$, the set $H^q_{\mathrm{cont}}(\widehat{G}; M)$ is countable. If $H_q(G; \mathbb{F}_p)$ is infinite-dimensional, the comparison (1.1) with trivial coefficients $\mathbb{F}_p$ is not surjective.*

Proof. There are countably many isomorphism types of finite groups. If G has d generators, there are at most $|F|^d$ homomorphisms to a fixed finite group F. Thus $\widehat{G}$ has only countably many open normal subgroups. Every continuous map $(\widehat{G})^q \to M$ factors through $(\widehat{G}/U)^q$ for some open normal U. Indeed, compactness gives a finite cover by product neighborhoods on which the map is constant; one can refine all coordinate neighborhoods by cosets of a common open normal subgroup. Intersecting with the kernel of the action if necessary gives such a U for the coefficient action as well. Each finite quotient has only finitely many cochains of degree q. The continuous cochain group, and consequently its cohomology, is therefore countable.

Over a field, algebraic duality is exact, so the universal coefficient isomorphism gives

$$H^q(G; \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}(H_q(G; \mathbb{F}_p), \mathbb{F}_p).$$

An infinite-dimensional vector space contains a countably infinite independent set. Arbitrary $\mathbb{F}_p$-valued sequences on that set extend to linear functionals on the whole vector space. Its dual thus has at least $p^{\aleph_0} = 2^{\aleph_0}$ elements. A map from the countable continuous cohomology group cannot be onto. □

Schreier rewriting gives a finite generating set for the kernels in (1.2).

Lemma 2.2. *The group S_k is generated by*

$$p_i = b_i a_i^{-1} \ (1 \leq i \leq k), \qquad q_i = a_i a_1^{-1} \ (2 \leq i \leq k).$$

Proof. Apply Schreier's rewriting with representatives a_1^n, $n \in \mathbb{Z}$, for the cosets of the kernel. For $i \geq 2$, the Schreier generators associated to a_i, b_i are $q_i, p_i q_i$, since the ith factor commutes with a_1. The generators associated to b_1 are $a_1^n p_1 a_1^{-n}$. Since a_2 commutes with the first factor,

$$q_2 p_1 q_2^{-1} = a_1^{-1} p_1 a_1.$$

Every one of these remaining generators is therefore a conjugate of p_1 by a power of q_2. The generators associated to a_1 are trivial, proving the assertion. □

Let $R_i = S^1 \vee S^1$, with its two loops labeled a_i, b_i. The product $T = R_1 \times \cdots \times R_k$ is a finite $K(A_k, 1)$: its universal cover is a product of trees. The cover T_χ associated to S_k is consequently a $K(S_k, 1)$. Set $\Lambda = \mathbb{F}_p[t, t^{-1}]$. Its cellular chain complex as an infinite cyclic cover is

$$C_*(T_\chi; \mathbb{F}_p) = \bigotimes_{i=1}^{k} K_i, \qquad K_i = [\Lambda e_{a_i} \oplus \Lambda e_{b_i} \xrightarrow{d} \Lambda], \tag{2.1}$$

where the nonzero terms of K_i are in degrees one and zero and $d(e_{a_i}) = d(e_{b_i}) = t - 1$. The product cellular resolution of A_k gives (2.1) after substituting the common deck generator t for every a_i, b_i.

Proposition 2.3. *For every prime p,*

$$H_k(S_k; \mathbb{F}_p) = \Lambda\,(u_1 \otimes \cdots \otimes u_k), \qquad u_i = e_{a_i} - e_{b_i}.$$

Consequently $H^k(S_k; \mathbb{F}_p) \cong \mathbb{F}_p^{\mathbb{Z}}$, with deck translation acting by a bilateral shift, up to inversion of the chosen generator.

Proof. Put $v_i = e_{b_i}$. Then (u_i, v_i) is a basis of $K_{i,1}$, $d(u_i) = 0$ and $d(v_i) = t - 1$. Write a top chain in the resulting tensor basis. In its boundary, the summand in which the ith factor has degree zero receives contributions only from basis tensors with v_i in that position. Its coefficients are the corresponding top coefficients multiplied, up to sign, by $t - 1$. Since Λ is an integral domain, vanishing of this boundary component forces all those coefficients to vanish. Doing this for every i leaves exactly the multiple of $u_1 \otimes \cdots \otimes u_k$. There are no chains in degree $k+1$, so no top boundaries.

The $\mathbb{F}_p$-basis of this homology is given by all Laurent translates $t^n(u_1 \otimes \cdots \otimes u_k)$, $n \in \mathbb{Z}$. Exact algebraic duality identifies the cohomology with the full space of functions on this basis. Deck translation permutes the basis by a shift, and its dual is the corresponding bilateral shift on sequences. □

Lemma 2.1, Lemma 2.2 and Proposition 2.3 prove that S_k is not good. The action of the cyclic quotient determines the comparison image.

Proposition 2.4. *The comparison image in $H^k(S_k; \mathbb{F}_p) \cong \mathbb{F}_p^{\mathbb{Z}}$ is exactly the union, over $d \geq 1$, of the d-periodic sequences.*

Proof. Let τ be conjugation by a_1 on S_k, and also its extension to $\widehat{S}_k$. The induced action on the cohomology in Proposition 2.3 is a shift or its inverse. A continuous cohomology class is represented by a cochain factoring through a finite quotient $\widehat{S}_k/U$. A topologically finitely generated profinite group has finitely many open subgroups of any fixed index: each such subgroup gives a continuous permutation representation of fixed finite degree, and there are finitely many such representations. Hence the orbit of U under τ is finite. Its intersection V is open, normal and τ-invariant. The cochain factors through $\widehat{S}_k/V$, and the automorphism induced by τ on this finite group has finite order, say d. The original cohomology class is τ^d-invariant. Its comparison image is therefore a periodic sequence.

Conversely, suppose a class $c \in H^k(S_k; \mathbb{F}_p)$ is invariant under τ^d. Let

$$A_{k,d} = \chi^{-1}(d\mathbb{Z}) = S_k \rtimes \langle a_1^d \rangle.$$

The Lyndon–Hochschild–Serre spectral sequence for this extension has only columns zero and one, since the quotient is infinite cyclic. Its edge map is surjective:

$$H^k(A_{k,d}; \mathbb{F}_p) \longrightarrow H^k(S_k; \mathbb{F}_p)^{\tau^d} \longrightarrow 0. \tag{2.2}$$

Thus c is the restriction of a class on $A_{k,d}$.

We use the classical goodness of finite products of finitely generated free groups; for instance this is a special case of [5, Theorem 3.15]. Goodness passes to finite-index subgroups: the profinite topology induced on a finite-index subgroup $H \leq G$ is its full profinite topology: the core in G of any finite-index subgroup of H is finite index in G. The discrete and continuous Shapiro isomorphisms, applied to the finite induced module of any finite H-module, identify its comparison with that for G. Hence $A_{k,d}$ is good. Lift the class supplied by (2.2) to $H^k_{\mathrm{cont}}(\widehat{A}_{k,d}; \mathbb{F}_p)$ and restrict along the continuous homomorphism $\widehat{S}_k \to \widehat{A}_{k,d}$. Naturality of restriction gives a continuous class whose comparison is c. □

For example the sequence supported only at $0 \in \mathbb{Z}$ is not periodic and gives an explicit class outside the comparison image.

3. Hyperplane and line-arrangement realizations

For $k \geq 3$ and $a \in \mathbb{C} \setminus \{0, 1\}$, put

$$Q_{k,a}(z_0, \ldots, z_{k-1}) = \left(\prod_{i=0}^{k-1} z_i \right) (az_0 - z_1) \left(\prod_{i=1}^{k-2} (z_i - z_{i+1}) \right) (z_{k-1} - z_0). \tag{3.1}$$

Its $2k$ linear factors are distinct. The projective complement

$$U_{k,a} = \mathbb{P}^{k-1} \setminus \{Q_{k,a} = 0\}$$

has fundamental group S_k. This is the specialization to free factors of rank two of [1, Lemmas 4.3 and 4.5]. Those lemmas identify the general fiber of the product map

$$f : (\mathbb{C}^* \setminus \{1\})^k \longrightarrow \mathbb{C}^*, \qquad f(x_0, \dots, x_{k-1}) = \prod_{i=0}^{k-1} x_i, \tag{3.2}$$

with the kernel of the induced character on the product of the free groups. The map is locally trivial away from 1, so the identification holds for all the stated values of a.

The fiber and the hyperplane complement are related by the explicit algebraic isomorphism

$$[z_0 : \cdots : z_{k-1}] \longmapsto \left(\frac{az_0}{z_1}, \frac{z_1}{z_2}, \dots, \frac{z_{k-2}}{z_{k-1}}, \frac{z_{k-1}}{z_0} \right). \tag{3.3}$$

All coordinates are nonzero and different from one exactly when the factors in (3.1) do not vanish, and their product is a. Conversely, normalize $z_0 = 1$ and set $z_j = x_j x_{j+1} \cdots x_{k-1}$ for $1 \le j \le k-1$. These formulas give the inverse. In a factor of $\mathbb{C}^* \setminus \{1\}$, loops α, β about $0, 1$ have character values $1, 0$. Replacing this free basis by $\alpha, \alpha\beta$ gives values $1, 1$ and hence exactly the character used in (1.2).

Since the coordinate hyperplane $z_0 = 0$ is deleted, the same projective complement is the complement of $2k-1$ affine hyperplanes in the chart $z_0 = 1$. The central complement

$$M_{k,a} = \mathbb{C}^k \setminus \{Q_{k,a} = 0\}$$

is isomorphic to $\mathbb{C}^* \times U_{k,a}$: use the coordinate z_0 and normalize the remaining coordinates by it. Thus its group is $S_k \times \mathbb{Z}$. The projection onto and inclusion of the first factor show that $H_k(S_k; \mathbb{F}_p)$ injects into $H_k(S_k \times \mathbb{Z}; \mathbb{F}_p)$. The latter is infinite-dimensional, so Lemma 2.1 proves nongoodness here as well.

For $k = 3$ and $a = 2$, the homogeneous equation is

$$z_0 z_1 z_2 (2z_0 - z_1)(z_1 - z_2)(z_2 - z_0) = 0.$$

Its affine chart gives exactly (1.3). This proves all the geometric assertions in Theorem 1.2. Higher members of the same family can also be realized by projective line arrangements, using generic plane sections which preserve the fundamental group; this is included in [1, Corollary 3.3].

Remark 3.1. There are also graphic central examples. Take a wheel with a hub and $m \ge 4$ rim vertices. Subtract the hub coordinate from each rim coordinate. The essential graphic arrangement then has equation

$$\left(\prod_{i=0}^{m-1} z_i \right) \left(\prod_{i=0}^{m-1} (z_i - z_{i+1}) \right) = 0, \qquad z_m = z_0.$$

This is (3.1) with $k = m$ and $a = 1$, up to a nonzero scalar. For $m \ge 4$, [1, Remark 4.6] states that the group at this special fiber agrees with the general-fiber group. Restoring the translation coordinate contributes a contractible factor, so the graphic complement has group $S_m \times \mathbb{Z}$ and is not good.

Let X be the five-line complement. It has the homotopy type of a finite two-dimensional CW-complex by the arrangement minimality theorem [3, Corollary 6]. Its Betti numbers are $b_1 = 5$, $b_2 = 7$, and $\chi(X) = 3$. For the second number one can use the affine intersection formula: there is one triple point and five double points, contributing $2 + 5 = 7$. The group cohomology $H^3(\pi_1 X; \mathbb{F}_p) = H^3(S_3; \mathbb{F}_p)$ is uncountable, whereas $H^3(X; \mathbb{F}_p) = 0$. In particular X is not aspherical. We have not yet resolved the arrangement-goodness question in the aspherical case.

4. The Euler identity and abstract coefficient groups

For a finite module M on a finite two-dimensional complex X, every cellular cochain group is finite. Taking orders in kernels and images gives

$$\frac{|H^0(X;M)|\,|H^2(X;M)|}{|H^1(X;M)|} = |M|^{\chi(X)}. \tag{4.1}$$

Indeed, the alternating product of cohomology orders equals that of cochain orders, and $C^j(X;M)$ is a product of one copy of M for each j-cell.

Lemma 4.1. *For any connected CW-complex with group G and any finite module M, $H^0(X;M)$ and $H^1(X;M)$ identify canonically with $H^0_{\mathrm{cont}}(\widehat{G};M)$ and $H^1_{\mathrm{cont}}(\widehat{G};M)$.*

Proof. In degree zero the space, group and continuous cohomology groups are the invariant elements of M. Density identifies invariants under G and $\widehat{G}$. In degree one space cohomology equals group cohomology, since the universal cover is simply connected: the low-degree sequence for $\widetilde{X} \to X \to BG$ uses $H^1(\widetilde{X};M) = 0$. A crossed homomorphism $d : G \to M$, satisfying

$$d(gh) = d(g) + g\,d(h),$$

defines a group homomorphism $g \mapsto (d(g), \rho(g))$ to the finite semidirect product $M \rtimes \mathrm{Aut}(M)$. It extends uniquely to $\widehat{G}$. Its first coordinate is a continuous crossed homomorphism with the extended action. Restriction is inverse to this extension and identifies principal crossed homomorphisms. Taking their quotient proves the canonical degree-one comparison. If $f : M \to N$ is a coefficient homomorphism, the extension of $f \circ d$ is f composed with the extension of d, by continuity and density. Principal derivations $g \mapsto gm - m$ are carried to $g \mapsto gf(m) - f(m)$, proving naturality in M. □

Proposition 4.2. *Under the hypotheses of Theorem 1.3, the groups $H^2(X;M^\phi)$ and $H^2(Y;M)$ are isomorphic as abstract finite abelian groups for every finite module M.*

Proof. Lemma 4.1 and (4.1) give equality of the orders for every finite coefficient module. More information comes from applying this to M/p^aM. Because there are no three-cells, H^2 is the cokernel of the degree-one coboundary. Right exactness of taking the coefficient quotient gives

$$H^2(X;M)/p^aH^2(X;M) \cong H^2(X;M/p^aM). \tag{4.2}$$

Indeed, if $\delta^1 : C^1(X;M) \to C^2(X;M)$ is the last coboundary, both sides are the quotient $C^2(X;M)/(\operatorname{im}\delta^1 + p^aC^2(X;M))$.

For a finite abelian group B, write its p-primary summand as $\bigoplus_i \mathbb{Z}/p^{e_i}\mathbb{Z}$. Then

$$\log_p |B/p^aB| = \sum_i \min(a, e_i).$$

The difference between consecutive values counts the summands with $e_i \geq a$. Thus these orders, for all primes and all a, determine the isomorphism type of B. Equation (4.2) and the already proved equality of orders give the result. □

The completed presentation module constructed next provides isomorphisms compatible with all coefficient homomorphisms.

5. Completed presentation modules and a natural comparison

Identify the completions using (1.4), and call the common group P. The completed integral group ring is the profinite ring

$$R = \widehat{\mathbb{Z}}[[P]] = \varprojlim_{n,U} (\mathbb{Z}/n\mathbb{Z})[P/U], \tag{5.1}$$

where U runs through the open normal subgroups of P. The kernels of these finite quotient maps are a basis of open two-sided ideals. Finite continuous left R-modules are exactly the objects of $\mathsf{Fin}(P)$: a finite module is annihilated by some n, and its continuous P-action factors through P/U for an open normal subgroup U, so the action factors through $(\mathbb{Z}/n\mathbb{Z})[P/U]$. Homomorphisms between profinite modules in this section are continuous unless explicitly stated otherwise. For a finite free left module E, write $E^* = \mathrm{Hom}_R(E, R)$; this is a right R-module.

Contract a maximal tree in each complex. We may assume that X has one vertex, s_X edges and r_X two-cells, and similarly for Y. In particular,

$$\chi(X) = 1 - s_X + r_X, \qquad \chi(Y) = 1 - s_Y + r_Y. \tag{5.2}$$

Choose lifts and orientations of the cells. Their presentation matrices, evaluated in R, are

$$R^{r_X} \xrightarrow{d_X} R^{s_X} \xrightarrow{e_X} R \xrightarrow{\varepsilon} \widehat{\mathbb{Z}}, \tag{5.3}$$

where e_X sends the edge generator corresponding to g_i to $g_i - 1$, and d_X is the cellular Fox matrix of the relators. The generators g_i have dense generated subgroup in P. In every finite quotient P/U, their images generate the whole group. The identity $gh - 1 = (g - 1) + g(h - 1)$ therefore shows that their augmentation differences generate the augmentation ideal of each finite quotient ring $(\mathbb{Z}/n\mathbb{Z})[P/U]$. These ideals are the images of $I = \ker \varepsilon$, so the image of e_X is dense in I. It is also closed, being the image of a compact module. Consequently e_X surjects onto I.

Exactness at R^{s_X} follows from the description of crossed homomorphisms with finite coefficients.

Lemma 5.1 (Exactness from finite derivations)**.** *The completed presentation is exact:*

$$R^{r_X} \xrightarrow{d_X} R^{s_X} \xrightarrow{e_X} I \longrightarrow 0.$$

Proof. Let $J = \mathrm{im}\, d_X$, a closed submodule since the domain is compact, and put $D = R^{s_X}/J$. The identity $e_X d_X = 0$ gives a continuous surjection $u : D \to I$.

For every finite left R-module M, assigning values to the edge generators extends uniquely to a crossed homomorphism on the free group. The Fox equations for the two-cell relators say exactly that this crossed homomorphism vanishes on the relators and descends to G_X. Indeed, every relator acts trivially on M, and the crossed-homomorphism identity gives $d(hrh^{-1}) = h\, d(r)$ for such a relator r. Vanishing on the relators therefore implies vanishing on their normal closure. Hence

$$\mathrm{Hom}_R(D, M) = \mathrm{Der}(G_X, M) = \mathrm{Der}_{\mathrm{cont}}(P, M). \tag{5.4}$$

The last equality is the finite semidirect-product extension from Lemma 4.1, before quotienting by principal derivations. Every map from the finite free module used here is continuous, since it is specified by finitely many values in a finite module.

We also have the augmentation-ideal identification

$$\mathrm{Hom}_R(I, M) = \mathrm{Der}_{\mathrm{cont}}(P, M). \tag{5.5}$$

To verify it directly, an R-linear map $h : I \to M$ gives the derivation $g \mapsto h(g - 1)$. Conversely a continuous derivation d and the action together give a continuous map to the finite semidirect product. They therefore factor through a common finite quotient Q of P. Choose n annihilating M. On the augmentation ideal of $(\mathbb{Z}/n\mathbb{Z})[Q]$, set $q - 1 \mapsto d(q)$. The elements $q - 1$ for $q \neq 1$ form a free $(\mathbb{Z}/n\mathbb{Z})$-basis of that ideal, and the crossed-homomorphism identity makes the map linear for the group-ring action: the identity $g(q - 1) = (gq - 1) - (g - 1)$ corresponds to $d(gq) - d(g) = g\, d(q)$. Composing with I gives a continuous R-linear map. Uniqueness follows because the elements $g - 1$ span a dense submodule of I.

Under (5.4) and (5.5), precomposition by u is the identity on derivations. It is therefore a bijection $\mathrm{Hom}_R(I, M) \to \mathrm{Hom}_R(D, M)$ for every finite M. If $0 \neq z \in \ker u$, choose a lift in R^{s_X}. Since J is

closed, there is an open two-sided ideal L such that the lift is not in $J + L^{s_X}$. Its image in the finite left module

$$D/LD = R^{s_X}/(J + L^{s_X})$$

is nonzero. The quotient map to this finite module cannot factor through u, contrary to the Hom bijection. Thus u is injective, which proves exactness. □

Define the compact right module

$$B_X = \operatorname{coker}(d_X^* : (R^{s_X})^* \to (R^{r_X})^*). \tag{5.6}$$

Its defining image is closed. Evaluation on a finite free module gives the isomorphism $E^* \otimes_R M \to \mathrm{Hom}_R(E, M)$ that sends $f \otimes m$ to the map $e \mapsto f(e)m$. Right exactness of tensor product, together with the fact that the degree-two cohomology is the final cokernel, therefore gives

$$H^2(X; M) = B_X \otimes_R M \tag{5.7}$$

naturally in each finite module M. The tensor product here is the ordinary algebraic tensor product. If L annihilates M, it is equally $(B_X/B_X L) \otimes_{R/L} M$, so it is determined at a finite ring level. The submodule $B_X L$ is closed: its preimage in the finite free module in (5.6) is $\operatorname{im} d_X^* + L^{r_X}$, a sum of compact submodules. Use the same construction for Y. In particular, $B_X/B_X L$ is finite, as a quotient of the finite module $(R/L)^{r_X}$.

Lemma 5.2 (Equal-rank stabilization)**.** *If $\chi(X) = \chi(Y)$, then for some $\ell \geq 0$ there is a continuous right R-module isomorphism*

$$B_X \oplus R^\ell \cong B_Y \oplus R^\ell. \tag{5.8}$$

Proof. Set $P_1 = R^{s_X}$, $P_2 = R^{r_X}$, $Q_1 = R^{s_Y}$ and $Q_2 = R^{r_Y}$. By Lemma 5.1, the maps $e_X : P_1 \to I$ and $e_Y : Q_1 \to I$ are surjective. Choose maps $a : P_1 \to Q_1$ and $b : Q_1 \to P_1$ with $e_Y a = e_X$ and $e_X b = e_Y$, by lifting the finitely many free basis elements. These maps are continuous.

On $T = P_1 \oplus Q_1$ put $e(x, y) = e_X x + e_Y y$ and $K = \ker e$. Two finite free modules surject onto K:

$$E_X = P_2 \oplus Q_1, \qquad j_X(u, v) = (d_X u - bv, v),$$
$$E_Y = Q_2 \oplus P_1, \qquad j_Y(u, v) = (v, d_Y u - av).$$

If $(x, y) \in K$, then $x + by \in \ker e_X$ and hence equals $d_X u$ for some u; thus $(x, y) = j_X(u, y)$. The argument for j_Y is identical with the two presentations interchanged.

Regard j_X, j_Y as maps into T. The cokernels of their dual maps $T^* \to E_X^*$ and $T^* \to E_Y^*$ are respectively B_X and B_Y. For the first, the dual has the form

$$(\lambda, \mu) \longmapsto (d_X^* \lambda, -b^* \lambda + \mu).$$

The arbitrary second coordinate eliminates the Q_1^* factor without adding a relation in the first; the remaining cokernel is (5.6). The other calculation is the same.

Define $j : E_X \oplus E_Y \to T$ by $j(e, f) = j_X e + j_Y f$. Since E_Y is finite free and j_X surjects onto K, there is a continuous map $A : E_Y \to E_X$ with $j_X A = j_Y$. The triangular change of basis $(e, f) \mapsto (e + Af, f)$ identifies j with $(j_X, 0)$. Dualizing and taking cokernels gives $\operatorname{coker} j^* \cong B_X \oplus E_Y^*$. Interchanging X, Y similarly gives $\operatorname{coker} j^* \cong B_Y \oplus E_X^*$. Hence

$$B_X \oplus E_Y^* \cong B_Y \oplus E_X^*.$$

Equation (5.2) says $r_X + s_Y = r_Y + s_X = \ell$ precisely when the Euler characteristics agree. Both extra free modules have this rank, giving (5.8). □

Lemma 5.3 (Cancellation over a finite ring)**.** *If A is a finite ring and B, C are finite right A-modules, then $B \oplus A^\ell \cong C \oplus A^\ell$ implies $B \cong C$.*

Proof. Every finite module has finite length, and successive decompositions give a finite direct sum of indecomposable modules. For an endomorphism f of an indecomposable finite module U, the kernels and images of sufficiently high powers stabilize. Choose n with $\ker f^n = \ker f^{2n}$ and $\operatorname{im} f^n = \operatorname{im} f^{2n}$. If $f^n u \in \ker f^n$, then $u \in \ker f^{2n} = \ker f^n$, so $f^n u = 0$. Thus $\ker f^n \cap \operatorname{im} f^n = 0$. The first isomorphism theorem gives $|U| = |\ker f^n|\,|\operatorname{im} f^n|$, and hence

$$U = \ker f^n \oplus \operatorname{im} f^n$$

for every sufficiently large n. Indecomposability implies that f is either invertible or nilpotent. In particular for each f, either f or $1 - f$ is invertible. This is the local-ring criterion for $\mathrm{End}_A(U)$: its nonunits form its unique maximal ideal. To see the criterion in this finite ring, two distinct maximal left ideals would give $x + y = 1$ with neither x nor y a unit; in a finite ring a left inverse is a two-sided inverse. Every nonunit belongs to a maximal left ideal, so uniqueness makes the set of nonunits closed under addition. It is also closed under multiplication on either side by arbitrary elements, since a product that is a unit has both factors invertible in a finite ring.

We recall the resulting uniqueness argument. Given two indecomposable decompositions of a module and a summand U of the first, compose its inclusion and projection with the projections and inclusions of the summands V_j of the second. The resulting endomorphisms of U sum to its identity. Since its endomorphism ring is local, one such composite is a unit. The corresponding maps $U \to V_j \to U$ show that U is a nonzero direct summand of V_j, and thus is isomorphic to V_j. The projection onto V_j restricts to an isomorphism on U, so the ambient module is also the direct sum of U and the remaining V-summands. Taking the quotient by U and proceeding inductively proves uniqueness of the multiset of indecomposable summands. Apply this to the two decompositions of $B \oplus A^\ell$ and $C \oplus A^\ell$, and cancel the common indecomposable summands of A^ℓ. □

Proposition 5.4 (An unstabilized completed isomorphism)**.** *There exists a continuous right R-module isomorphism $B_X \cong B_Y$.*

Proof. For each open two-sided ideal L in the quotient basis of R, let $R_L = R/L$ and $B_{X,L} = B_X/B_X L$, and similarly for Y. These are finite modules. Reducing (5.8) modulo L and applying Lemma 5.3 shows that

$$\mathcal{S}_L = \mathrm{Iso}_{R_L}(B_{X,L}, B_{Y,L})$$

is a nonempty finite set. If $L' \subseteq L$, every isomorphism at level L' descends to one at level L, by tensoring with R_L. These transition maps define an inverse system.

Its inverse limit is nonempty. In the compact product $\prod_L \mathcal{S}_L$, each compatibility equation defines a closed subset. A finite family of such equations is satisfied by choosing a common refinement of all involved ideals, choosing an isomorphism there, and projecting to those levels. Fill the unconstrained coordinates with arbitrary elements. The finite intersection property and compactness give a family of isomorphisms compatible at all levels.

Finally $B_X \cong \varprojlim_L B_{X,L}$ and similarly for Y. Indeed, writing $B_X = R^{r_X}/J_X$, the preimages $J_X + L^{r_X}$ form a neighborhood basis of the closed submodule J_X. Thus the submodules $B_X L$ form a neighborhood basis at zero and have zero intersection. A compatible family in the finite quotients specifies closed cosets in B_X with the finite intersection property. Compactness gives a representative in B_X, proving surjectivity to the inverse limit; the zero intersection proves injectivity. Taking the inverse limit of the compatible isomorphisms proves the proposition. □

Proof of the cohomology assertion in Theorem 1.3. Choose an isomorphism in Proposition 5.4 and tensor it with every finite left R-module M. Formula (5.7) gives the degree-two transformation. The same chosen module map is used for every M, so all maps of coefficient modules commute with it. Explicitly, if $\eta : B_X \to B_Y$ is the chosen isomorphism and $f : M \to N$ is a coefficient homomorphism, then

$$(\eta \otimes 1_N)(1_{B_X} \otimes f) = (1_{B_Y} \otimes f)(\eta \otimes 1_M).$$

Degrees zero and one are Lemma 4.1; higher degrees vanish by dimension. The choices in the completed-module isomorphism explain why the natural transformation in degree two is noncanonical. □

Remark 5.5. Finite local homology also admits a simultaneous natural comparison. Give $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ the left action $(g\lambda)(m) = \lambda(g^{-1}m)$. For local homology, regard the left cellular chains of the universal cover as right chains by the involution $g \mapsto g^{-1}$ and tensor with the left module M. Evaluation identifies the dual of this chain complex with the cellular cochain complex with coefficients $M^\vee$. Since Pontryagin duality is exact on finite groups, it gives $H_q(X; M)^\vee \cong H^q(X; M^\vee)$. Apply (1.5) to $M^\vee$ and dualize. The natural finite bidual identification $M \cong M^{\vee\vee}$ yields a comparison covariant in M.

6. Integral homology of every finite cover

Integral homology of a finite two-dimensional cover is determined by its first homology and Euler characteristic.

Lemma 6.1. *The profinite completion of a finitely generated group determines its abelianization as an abstract finitely generated abelian group. The completion of every finite-index subgroup is its closure in the original completion.*

Proof. Finite abelian quotients show that $\widehat{G_{\mathrm{ab}}}$ is the maximal abelian profinite quotient of $\widehat{G}$. If $G_{\mathrm{ab}} = \mathbb{Z}^b \oplus T$ with T finite, its pro-p abelian quotient is $\mathbb{Z}_p^b \oplus T_p$. The finite torsion subgroup is exactly T_p, and the torsion-free $\mathbb{Z}_p$-rank is b. All primary summands of T and the rank are therefore determined.

For the second assertion, if $N \le G$ is finite index and $L \le N$ is finite index, the core of L in G is a finite-index normal subgroup of G contained in L. Thus the induced profinite topology on N is full. Its completion maps injectively onto the closure of N, which is the open subgroup corresponding to N. This proof does not require G to be residually finite. □

Proof of the finite-cover assertion in Theorem 1.3. For an open subgroup $U \le P$, pull it back under the two completion maps to subgroups $N_X \le G_X$ and $N_Y \le G_Y$. Their indices are the same finite number $d = [P : U]$, since the images of the discrete groups are dense. By Lemma 6.1, their completions are both U. They are finitely generated by Schreier's theorem, so the same lemma gives

$$H_1(X_U; \mathbb{Z}) = N_{X,\mathrm{ab}} \cong N_{Y,\mathrm{ab}} = H_1(Y_U; \mathbb{Z}).$$

Both covers are connected, with $H_0 = \mathbb{Z}$. Their Euler characteristics are $d\chi(X)$ and $d\chi(Y)$. For a finite two-dimensional complex, $H_2(-; \mathbb{Z})$ is a subgroup of its finite free group of two-chains, hence is finite free abelian, with rank

$$b_2 = d\chi(X) + b_1 - 1.$$

The ranks agree. There is no homology above degree two, which proves (1.6). Applying the finite-coefficient part of the theorem to these covers gives the corresponding natural finite-coefficient conclusion as well. □

The isomorphisms in (1.6) are isomorphisms of abelian groups. The finite-coefficient comparison is natural in the coefficients on each corresponding pair of covers.

7. Recovering the Euler characteristic for line arrangements

For spaces whose rational cohomology in degree two is generated by cup products, the completion also determines the Euler characteristic needed in Theorem 1.3. Write $\gamma_1 G = G$ and $\gamma_{i+1} G = [G, \gamma_i G]$. The second lower central quotient is $\mathrm{gr}_2 G = \gamma_2 G / \gamma_3 G$.

Lemma 7.1. *Let X be a finite two-dimensional complex and $G = \pi_1(X)$. If the rational cup product*

$$\bigwedge^2 H^1(X;\mathbb{Q}) \longrightarrow H^2(X;\mathbb{Q}) \tag{7.1}$$

is surjective, then

$$\chi(X) = 1 - b_1(G) + \binom{b_1(G)}{2} - \operatorname{rank}(\gamma_2 G/\gamma_3 G). \tag{7.2}$$

Consequently $\chi(X)$ is determined by $\widehat{G}$ within this class of spaces.

Proof. The classifying map $X \to BG$ induces an isomorphism in degree one and an injection $H^2(G;\mathbb{Q}) \hookrightarrow H^2(X;\mathbb{Q})$. For the latter, the low-degree sequence of the universal-cover fibration uses $H^1(\widetilde{X};\mathbb{Q}) = 0$. Naturality of cup product shows that the image of this injection contains the image of (7.1). Surjectivity of (7.1) therefore makes it an isomorphism and makes the group cup product $\bigwedge^2 H^1(G;\mathbb{Q}) \to H^2(G;\mathbb{Q})$ surjective.

The homology five-term sequence of $1 \to \gamma_2 G \to G \to G_{\mathrm{ab}} \to 1$ gives

$$H_2(G;\mathbb{Q}) \longrightarrow \bigwedge^2 H_1(G;\mathbb{Q}) \longrightarrow (\gamma_2 G/\gamma_3 G) \otimes \mathbb{Q} \longrightarrow 0. \tag{7.3}$$

The first map is dual to the group cup product, and hence is injective. Since $H^2(G;\mathbb{Q}) = H^2(X;\mathbb{Q})$ is finite-dimensional, exactness gives

$$b_2(X) = \binom{b_1(G)}{2} - \operatorname{rank}(\gamma_2 G/\gamma_3 G).$$

The Euler formula in dimension two proves (7.2).

To recover the rank of the lower central quotient, let $P = \widehat{G}$ and let $\overline{\gamma_3(P)}$ denote the closed subgroup generated by commutators of weight three, and set $N = G/\gamma_3 G$. By density and continuity of word maps, the closure of the image of $\gamma_3 G$ in P is $\overline{\gamma_3(P)}$. The universal property of completion therefore gives

$$\widehat{N} = P/\overline{\gamma_3(P)}.$$

The group $C = [N,N] = \gamma_2 G/\gamma_3 G$ is finitely generated abelian and central in the finitely generated nilpotent group N. If $g_1,\dots,g_s$ generate N, the central commutators $[g_i,g_j]$ generate C. The profinite topology induced on C by N is full. Indeed, for every finite-index subgroup $D \le C$, the quotient N/D is finitely generated nilpotent, hence residually finite by the classical theorem of Mal'cev [6]. Separate each of the finitely many nonidentity elements of C/D in a finite quotient of N/D and take the product of these quotients. The resulting finite-index kernel in N intersects C in D. Thus $\widehat{C}$ embeds in $\widehat{N}$ as the closure of C.

That closure is the closed commutator subgroup of $\widehat{N}$: commutator values on the dense image of N are dense among commutator values on $\widehat{N}$. It is therefore intrinsically determined by P. Lemma 6.1, applied to the finitely generated abelian group C, recovers its rank from this closed subgroup. Abelianization recovers $b_1(G)$ by the same lemma. This proves the assertion. The argument is the degree-two instance of the more general graded-quotient recovery in [9, Theorem C]. □

Proof of Corollary 1.4. An affine line-arrangement complement has the homotopy type of a finite two-dimensional complex. The same holds for a nonempty projective line arrangement, by taking a deleted arrangement line as the line at infinity. These facts follow from the arrangement minimality theorem [3, Corollary 6], whose construction has cells only up to the complex dimension of the complement. The Orlik–Solomon theorem [7] identifies the integral cohomology algebra of the affine complement with an exterior algebra modulo the arrangement relations. In particular it is

generated in degree one. Thus the rational cup product (7.1) is surjective for each complement. Apply Lemma 7.1 to recover the common Euler characteristic, and then apply Theorem 1.3. □

For the five-line complement, S_3 fails goodness in degree three, and its completion determines the abstract integral homology of every finite cover. Corollary 1.4 also gives a simultaneous coefficient-natural comparison with every line-arrangement complement having the same completion.

Remark 7.2 (Dimension, Euler characteristic and space maps). Without the equal-Euler hypothesis in the general two-dimensional theorem, a point and S^2 have the same trivial profinite fundamental group but different degree-two cohomology. Without the dimension bound, a point and $S^2 \vee S^3$ have the same completion and the same Euler characteristic, but different cohomology. Finally, even for $X = Y = S^2$, the identity and a self-map of degree p induce the same isomorphism of profinite fundamental groups. On $H^2(S^2; \mathbb{F}_p)$ they induce respectively the identity and zero. Consequently the existence of a coefficient-natural isomorphism cannot be strengthened to recovery of the action of every space map from its action on the completion.

The module recovered in the proof is the compact right module $B_X = \operatorname{coker} d_X^*$ over R. The second homotopy module has the different presentation $\pi_2(X) = \ker(d_2 : \mathbb{Z}G_X^{r_X} \to \mathbb{Z}G_X^{s_X})$. Theorem 1.3 recovers the former module through its finite quotient modules and their tensor products with finite coefficients.

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