# ALEXANDER INVARIANTS AND FINITE ABELIAN COVERS

## METABELIAN SEPARATION FOR GRAPHIC ARRANGEMENTS AND THE ARBITRARY-WEIGHT MILNOR MONODROMY QUESTION

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Abstract. We determine the actual integral Alexander invariant of every finite simple graphic arrangement. Natural maps to the Alexander invariant of a rank-two free group give an injection into the sum of the two-variable Laurent rings indexed by triangles and four-cliques. The cokernel is a trivial free abelian module of rank equal to the number of four-cliques; its defining equations are explicit augmentation compatibilities. The augmentation filtration is strict. Consequently every graphic metabelian quotient is residually torsion-free nilpotent, answering the question recorded in Suciu's Remark 8.7 and extending its stated range. We then compute the complete first-homology representation of every finite abelian cover over fields of characteristic prime to the covering degree. This yields a gcd formula for the rational monodromy of every primitively weighted graphic Milnor fiber, and excludes torsion primes not dividing its degree. A direct rank-three example gives a negative answer to the arbitrary-weight monodromy questions in the checked arXiv version of Suciu's Question 12.3. All group, module, and covering arguments required for these conclusions are included.

## 1. The questions and the structural answer

The Alexander invariant of a group $G$ with free abelian abelianization is the module $B(G) = G'/G''$ over $\mathbb{Z}[G^{\mathrm{ab}}]$, with the action induced by conjugation. Its augmentation filtration computes the lower central series of the maximal metabelian quotient. Controlling the associated graded module does not, by itself, show that the uncompleted module is separated. This distinction is essential in the question of Suciu [8, Remark 8.7]: for a $K_4$-free graphic arrangement group, is its metabelian quotient residually nilpotent?

We answer this question by determining the uncompleted module, integrally, for every finite simple graph. A second question concerns weighted Milnor fibers. The two arbitrary-weight questions in [8, Question 12.3] ask for trivial integral, respectively rational, monodromy under decomposability hypotheses. The polynomial $xy(x-y)z^3$ gives a negative answer to both. The positive replacement is an explicit arithmetic formula for all graphic arrangements and all primitive positive weights.

Fix a finite simple graph $\Gamma = (V, E)$ with its vertices totally ordered. Write $e = |E|$, and set

$$M_\Gamma = \{z \in \mathbb{C}^V : z_i \neq z_j \text{ for } ij \in E\}, \qquad G_\Gamma = \pi_1(M_\Gamma).$$

The base point has all coordinates distinct. Positive edge meridians are denoted $a_{ij}$. The abelianization is $\mathbb{Z}^E$, with basis $u_{ij}$, and

$$R_\Gamma = \mathbb{Z}[t_{ij}^{\pm 1} : ij \in E], \qquad I_\Gamma = (t_{ij} - 1 : ij \in E), \qquad B_\Gamma = G'_\Gamma/G''_\Gamma.$$

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We use $[a,b]=aba^{-1}b^{-1}$ and write the abelian group $B_\Gamma$ additively. Thus conjugation by $a_{ij}$ acts as multiplication by $t_{ij}$. Let $\mathcal{C}_s$ be the set of $s$-vertex cliques of $\Gamma$, and $c_s=|\mathcal{C}_s|$. For $S\subseteq V$, $E(S)$ denotes the edges with both endpoints in $S$.

For $T=\{i<j<k\}\in\mathcal{C}_3$, define

$$J_T=(t_a-1:a\notin E(T))+(t_{ij}t_{ik}t_{jk}-1),\qquad R_T=R_\Gamma/J_T.$$

For $Q=\{i<j<k<\ell\}\in\mathcal{C}_4$, define

$$\begin{aligned}J_Q=&(t_a-1:a\notin E(Q))\\&+(t_{ij}-t_{k\ell},t_{ik}-t_{j\ell},t_{i\ell}-t_{jk},t_{ij}t_{ik}t_{jk}-1),\qquad R_Q=R_\Gamma/J_Q.\end{aligned}$$

Each quotient is a Laurent polynomial ring in two variables over $\mathbb{Z}$. Put $D_\Gamma=\bigoplus_{T\in\mathcal{C}_3}R_T\oplus\bigoplus_{Q\in\mathcal{C}_4}R_Q$. Its augmentation maps are denoted $\varepsilon_S$.

**Theorem 1.1** (Integral Alexander structure)**.** *There is a natural $R_\Gamma$-linear injection $\beta:B_\Gamma\longrightarrow D_\Gamma$. After the target bases specified in Section 4 are chosen, it fits into the exact sequence*

$$0\longrightarrow B_\Gamma\xrightarrow{\beta}D_\Gamma\xrightarrow{\lambda}\mathbb{Z}^{c_4}\longrightarrow 0, \tag{1}$$

*where $R_\Gamma$ acts on the last term by augmentation, and*

$$\lambda_Q((u_T)_T,(v_{Q'})_{Q'})=\varepsilon_Q(v_Q)-\sum_{r=1}^{4}(-1)^r\varepsilon_{Q\setminus\{q_r\}}(u_{Q\setminus\{q_r\}}) \tag{2}$$

*for $Q=\{q_1<q_2<q_3<q_4\}$. Identifying $B_\Gamma$ with its image,*

$$I_\Gamma^rB_\Gamma=I_\Gamma^rD_\Gamma\quad(r\geq 1),\qquad B_\Gamma\cap I_\Gamma^rD_\Gamma=I_\Gamma^rB_\Gamma\quad(r\geq 0). \tag{3}$$

*The exact sequence remains exact after tensoring over $\mathbb{Z}$ with any commutative coefficient ring.*

The explicit free-group maps are classical [2]; their simultaneous faithfulness and the integral cokernel calculation are proved below. The Chen numerical formulas obtained here recover classical computations [3, 7]; the additional information is the actual module, the natural detection maps, and strictness before completion. The rational separation perspective in [9] concerns an infinitesimal comparison. Here the integral argument works directly with the actual module and its detection maps.

## 2. A self-contained planar group comparison

Write $P_V=\pi_1\operatorname{Conf}_V(\mathbb{C})$ for the full pure braid group. For $T\subseteq V$, let $q_T:P_V\to P_T$ forget the other points; $P_\varnothing=1$. Put

$$J_i^{\mathrm{grp}}=\ker q_{V\setminus\{i\}},\qquad J_T^{\mathrm{grp}}=\bigcap_{i\in T}J_i^{\mathrm{grp}},\qquad \mathrm{Br}_T=\bigcap_{i\in T}\ker(P_T\to P_{T\setminus\{i\}}).$$

The superscript distinguishes group kernels from the Laurent ideals above.

**Lemma 2.1** (Two kernels and Brunnian lifts)**.** *For distinct vertices $i,j$, the normal closure of $a_{ij}$ in $P_V$ is $J_i^{\mathrm{grp}}\cap J_j^{\mathrm{grp}}$. For every nonempty $T$, there is a group homomorphism $s_T:P_T\to P_V$ such that $q_Ts_T=\mathrm{id}$ and $s_T(\mathrm{Br}_T)\subseteq J_T^{\mathrm{grp}}$.*

*Proof.* The full-configuration Fadell–Neuwirth fibration [4] identifies $J_j^{\mathrm{grp}}$ with the fundamental group of the plane punctured at the other base points. Its injection follows from the asphericity of the base, obtained inductively from the same fibration. Filling the $i$-th puncture kills precisely the normal closure of its peripheral loop. Consequently $J_i^{\mathrm{grp}}\cap J_j^{\mathrm{grp}}$ is the normal closure of $a_{ij}$ inside $J_j^{\mathrm{grp}}$. This intersection is normal in $P_V$. Its local normal closure is therefore both contained in, and contains, the normal closure in $P_V$, proving the first assertion.

For the second assertion, join the base points indexed by $T$ by an embedded tree avoiding all the others. A small smooth Jordan neighborhood $U_T$ is a disk containing precisely these base

points. The inclusion $\mathrm{Conf}_T(U_T) \to \mathrm{Conf}_T(\mathbb{C})$ is a based homotopy equivalence. Indeed, a plane homeomorphism taking $U_T$ to the unit disk reduces the claim to a radial compression fixing a smaller disk containing the selected points. Choose a strictly increasing radial function equal to the identity on that smaller disk and tending to 1 at infinity. Interpolating with the identity gives embeddings, fixes the base points, and preserves the unit disk. These maps induce the required homotopies of configuration spaces. The same construction applies after deleting any selected point. Keep all points outside $T$ fixed, and use the induced fundamental-group isomorphism to define $s_T$. If a braid is Brunnian, each single deletion is null-homotopic already inside $U_T$; keeping the outside points fixed preserves that null-homotopy. Thus its image is in every indicated kernel. No compatibility between different choices of $U_T$ is required. □

Let $N_\Gamma$ be the normal closure in $P_V$ of the nonedge meridians. The inclusion $\mathrm{Conf}_V(\mathbb{C}) \subseteq M_\Gamma$ gives

$$G_\Gamma = P_V/N_\Gamma. \tag{4}$$

For completeness, a loop in $M_\Gamma$ can be moved off the additionally allowed diagonals. A null-homotopy can be made transverse to their smooth codimension-two strata and disjoint from the strata of codimension at least four. Its finitely many intersection points contribute conjugates of precisely the nonedge meridians. Conversely these meridians contract after the corresponding collisions are allowed. This proves both surjectivity and the claimed kernel, without assuming $M_\Gamma$ aspherical.

**Proposition 2.2** (Clique detection and verbal strictness)**.** *The map $\rho : G_\Gamma \to \prod_{C \text{ maximal clique}} P_C$ is injective. If $W(H)$ is a verbal subgroup of $H$, then*

$$\rho(G_\Gamma) \cap \prod_C W(P_C) = \rho(W(G_\Gamma)).$$

*In particular, $G_\Gamma/G''_\Gamma$ injects into $\prod_C P_C/P''_C$.*

*Proof.* Order all nonempty subsets of $V$ by increasing cardinality, breaking ties arbitrarily. Starting with $h = g \in P_V$, suppose all earlier projections $q_S(h)$ have been killed. Then $q_T(h) \in \mathrm{Br}_T$, since its single deletions are earlier projections. Replace $h$ by $s_T(q_T(h))^{-1}h$. The correction belongs to $J_T^{\mathrm{grp}}$ and is killed by every earlier $q_S$: such $S$ cannot contain $T$, so some $i \in T \setminus S$ is deleted. The process terminates at $T = V$ and gives a finite factorization of $g$ into corrections in $J_T^{\mathrm{grp}}$.

If $g$ is killed by all clique projections, every correction for a clique $T$ is 1. To check this inductively, earlier nontrivial corrections have nonclique supports $U$, which cannot be contained in $T$, and hence have trivial $q_T$-image. Every nonclique $U$ contains a nonedge $ij$, and

$$J_U^{\mathrm{grp}} \subseteq J_i^{\mathrm{grp}} \cap J_j^{\mathrm{grp}} = \langle\langle a_{ij} \rangle\rangle \subseteq N_\Gamma.$$

Thus the common kernel of the maximal-clique projections is $N_\Gamma$. The reverse inclusion is immediate. Equation (4) proves injectivity.

For strictness, take an element $a \in G_\Gamma$ whose clique images lie in $W$. They define a compatible family $a_T \in W(P_T)$ on all cliques. Starting from $h = 1 \in W(P_V)$, process cliques in increasing cardinality. For the current $T$, set $d_T = q_T(h)^{-1}a_T$. Compatibility makes $d_T$ Brunnian, and both factors show $d_T \in W(P_T)$. Replace $h$ by $hs_T(d_T)$. A homomorphism preserves verbal subgroups, so $h$ remains in $W(P_V)$; the earlier projections remain matched by Lemma 2.1. At the end, the image of $h$ in $G_\Gamma$ has the same maximal-clique projections as $a$. Injectivity makes the two equal. The reverse containment is functorial. Taking $W(H) = H''$ proves the last assertion. □

The verbal-strictness argument above supplies the injectivity needed after metabelianization and completes the planar comparison within this note.

## 3. Actual four-strand detection

We first work with $P_n$, where $V = [n]$ is complete, and write $R_n, I_n, B_n$. The following exact Laurent-module input comes from [3, Theorem 3.4, Corollary 3.6 and the displayed relations on p. 56]. It is an input about $B_n$, before completion.

**Proposition 3.1** (Classical presentation input)**.** *The elements $b_T = [a_{ij}, a_{jk}]$, $T = \{i < j < k\}$, generate $B_n$ over $R_n$. For distinct $p, q \notin T$ and $i, j \in T$,*

$$(t_{pq} - 1)b_T = 0, \qquad (t_{ip} - 1)(t_{jq} - 1)b_T = 0. \tag{5}$$

The published relations first give these formulas for $T = \{1, 2, 3\}$ with $p, q > 3$. The extension to arbitrary $T$ needs attention because standard Artin generators are not freely permuted. The braid conjugation constructed in [2, Proposition 3.2] carries the three generators on an ordered selected triple to the corresponding initial-triple generators. On abelianization it permutes the unordered-pair variables. It therefore transports the actual annihilator formulas exactly. Alternatively a disk containing the selected triple transports its rank-one $P_3$ Alexander module to the initial disk; a Laurent-unit change of its basis does not change an annihilator. The triple generators in the cited presentation are the commutator cycles stated here, up to this fixed unit convention.

For each vertex $i$, deletion followed by the standard generator section defines an endomorphism $p_i : P_n \to P_n$: it kills incident generators and fixes the others. The generator section is justified by [1, Proposition 4.7]; one can also check the Artin relations directly. The $p_i$ are commuting idempotents. On $B_n$ they are additive idempotents, semilinear for the commuting ring maps $\sigma_i$ setting incident variables to 1. In particular,

$$p_i(b_T) = 0 \ (i \in T), \qquad p_i(b_T) = b_T \ (i \notin T).$$

**Lemma 3.2** (The full-support kernel vanishes)**.** *For $n \geq 5$, simultaneous single-strand deletion is injective on $B_n$. Consequently simultaneous four-strand deletion detects $B_n$ for every $n \geq 4$.*

*Proof.* Set $x_{ij} = t_{ij} - 1$ and $K_p = (x_{pj} : j \neq p)$. Then $R_n = \mathbb{Z}[x_{ij}][(1 + x_{ij})^{-1}]$ and $K_p = \ker \sigma_p$. For a vertex set $O$, $\bigcap_{p \in O} K_p$ is generated by monomials whose edge endpoints cover $O$. This is the termwise description of the intersection of coordinate monomial ideals in $\mathbb{Z}[x_{ij}]$; localization preserves the finite intersection.

If $O = [n] \setminus T$ has at least two vertices, each such monomial kills $b_T$. An edge entirely in $O$ supplies the first factor in (5). Otherwise choose distinct $p, q \in O$; the covering monomial contains $x_{ip}x_{jq}$ with $i, j \in T$, and the second formula in (5) applies.

The additive projection $E_n = \prod_{i=1}^{n}(1 - p_i)$ has image $\bigcap_i \ker p_i$. Semilinearity gives

$$E_n(fb_T) = \left( \prod_{i \notin T} (1 - \sigma_i) f \right) b_T.$$

The coefficient lies in every $K_p$ for $p \notin T$, since the ring maps commute. The preceding paragraph makes it zero on $b_T$ for $n \geq 5$. Generation gives $E_n B_n = 0$, hence the common deletion kernel is zero. Iterating single deletions proves the four-strand assertion. The argument works directly with integral uncompleted modules. □

3.1. **The integral four-strand calculation.** Let $R = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}, f^{\pm 1}]$, where $(a, b, c, d, e, f) = (t_{12}, t_{13}, t_{23}, t_{14}, t_{24}, t_{34})$. The specialized presentation of [3, Theorem 3.5, p. 55] uses the following explicit source-unit normalization:

$$u = b_{123}, \qquad v = b_{234}, \qquad w = t_{24}^{-1} b_{134}, \qquad z = -(t_{24}t_{34})^{-1} b_{124}.$$

With the chain-to-commutator convention fixed as above, it has the following fourteen relations:

$$\begin{aligned}
&(1-abc)u, \quad (1-def)u,\\
&(1-cef)v, \quad (1-f)u+(1-a)v, \quad (1-e)u+(1-b)v, \quad (1-d)(u+v),\\
&(1-bdf)w, \quad (f-1)u+(1-a)w, \quad (e-1)u+(1-e)w, \quad (d-1)u+(1-c)w,\\
&(1-ade)z, \quad (1-f)(u-z), \quad (1-e)u-(1-b)z, \quad (1-d)u-(1-c)z.
\end{aligned} \tag{6}$$

Here a displayed expression is a relation set equal to zero. Define

$$\begin{aligned}
L_1 &= (d-1, e-1, f-1, abc-1), & L_2 &= (a-1, b-1, d-1, cef-1),\\
L_3 &= (a-1, c-1, e-1, bdf-1), & L_4 &= (b-1, c-1, f-1, ade-1),\\
L_5 &= (d-c, e-b, f-a, abc-1).
\end{aligned}$$

Each $R/L_i$ is a two-variable Laurent ring. Direct substitution into (6) defines

$$\begin{aligned}
&\Phi : B_4 \longrightarrow \bigoplus_{i=1}^{5} R/L_i,\\
&u \longmapsto (1,0,0,0,1), \qquad v \longmapsto (0,1,0,0,-1),\\
&w \longmapsto (0,0,1,0,1), \qquad z \longmapsto (0,0,0,1,1).
\end{aligned} \tag{7}$$

**Lemma 3.3.** *The map $\Phi$ is injective and its cokernel is $R/I_4 = \mathbb{Z}$, where $I_4 = (a-1, b-1, c-1, d-1, e-1, f-1)$.*

*Proof.* Put $L = (abc-1, def-1)$, $U = (d-1, e-1, f-1)$, and $H = (d-c, e-b, f-a)$. We prove

$$\mathrm{Ann}_R(u) = L + UH = L_1 \cap L_5. \tag{8}$$

The first two relations give $Lu = 0$. Multiply the three mixed $v$ relations by $1-cef$ to obtain $(1-cef)Uu = 0$; the identity $d(1-cef) = (d-c) - c(def-1)$ then gives $(d-c)Uu = 0$. The $w$ and $z$ blocks similarly give $(1-bdf)Uu = 0$ and $(1-ade)Uu = 0$. Using $e(1-bdf) = (e-b) - b(def-1)$ and $f(1-ade) = (f-a) - a(def-1)$ gives the other two generators of $HUu = 0$. Thus $L+UH \subseteq \mathrm{Ann}(u)$. Well-definedness of $\Phi$ gives $\mathrm{Ann}(u) \subseteq L_1 \cap L_5$.

To identify that intersection, pass to $S = R/L = \mathbb{Z}[a^{\pm1}, b^{\pm1}, d^{\pm1}, e^{\pm1}]$, where $c = (ab)^{-1}$ and $f = (de)^{-1}$. The images of $U, H$ are respectively $(d-1, e-1)$ and $(r, s) = (d-c, e-b)$. The omitted third generators belong to these ideals by the two product identities. Modulo $U$, $r, s$ become $1-(ab)^{-1}, 1-b$, a regular sequence: the quotient by the first is $\mathbb{Z}[b^{\pm1}]$, in which the second is a non-zero-divisor. If $pr + qs \in U$, the two-term Koszul relation modulo $U$ gives $\bar{p} = -\bar{h}\bar{s}$ and $\bar{q} = \bar{h}\bar{r}$. Lift $\bar{h}$ to $h \in S$. Then $p + hs, q - hr \in U$ and $pr + qs = (p+hs)r + (q-hr)s \in UH$. Hence $U \cap H = UH$ in $S$; taking inverse images proves (8).

Setting $u = 0$ in (6) gives

$$B_4/Ru = (R/L_2)v \oplus (R/L_3)w \oplus (R/L_4)z.$$

For an element in $\ker\Phi$, coordinates $2, 3, 4$ therefore show that it is $pu$. Coordinates $1, 5$ give $p \in L_1 \cap L_5 = \mathrm{Ann}(u)$, so it vanishes. This proves injectivity.

In the cokernel, the four columns in (7) express the first four target generators in terms of the fifth. Its annihilator is $L_1 + \cdots + L_5 = I_4$: all ideals lie in $I_4$, and their generators already include all six differences from 1. This proves the cokernel assertion, and in particular $I_4 \bigoplus_i R/L_i \subseteq \mathrm{im}\,\Phi$. □

## 4. The natural maps and the global cokernel

Let $F = F(x, y)$ and $\kappa = [x, y] \in B(F)$. The two-circle universal abelian cover has boundary $(X-1, Y-1) : \mathbb{Z}[X^{\pm1}, Y^{\pm1}]^2 \to \mathbb{Z}[X^{\pm1}, Y^{\pm1}]$. Its kernel is freely generated by $(1-Y, X-1)$, so $B(F)$ is a free rank-one Laurent module generated by $\kappa$.

The classical epimorphisms of [2, eqs. (3.1)–(3.2)] have formulas

$$f_T : (a_{ij}, a_{ik}, a_{jk}) \longmapsto (x, y, y^{-1}x^{-1}),$$
$$f_Q : (a_{ij}, a_{ik}, a_{jk}, a_{i\ell}, a_{j\ell}, a_{k\ell}) \longmapsto (x, y, y^{-1}x^{-1}, y^{-1}x^{-1}, xyx^{-1}, x). \tag{9}$$

All unspecified meridians go to 1. For a clique these maps factor through $G_\Gamma$ by (4), and they are onto because $x, y$ occur. Their scalar kernels are $J_T, J_Q$. Choose the basis $-Y^{-1}\kappa$ in each target, so that $\beta_T(b_T) = 1$ and $\beta_Q(b_{q_1q_2q_3}) = 1$. These choices define $\beta$.

On four strands the four triangle coordinates identify, up to Laurent units, with coordinates $1, 2, 3, 4$ of $\Phi$. For example,

$$[x, y^{-1}x^{-1}] = -Y^{-1}\kappa, \quad [y^{-1}x^{-1}, x] = Y^{-1}\kappa, \quad [y, x] = -\kappa, \quad [x, xyx^{-1}] = X\kappa.$$

Applying these identities to the source-unit changes in (6) makes the fifth row a unit multiple of $(1, -1, 1, 1)$. Thus Lemma 3.3 concerns precisely the five actual group-induced Alexander maps. Lemma 3.2 now implies that all triangle and four-clique maps detect $B_n$.

To pass to a graph, Proposition 2.2 embeds its metabelian quotient into those of its maximal cliques. Apply the just-proved detection to each complete-clique factor. The maps in (9) agree with deletion followed by the corresponding maps on that clique; duplicate maps can be discarded. This proves injectivity of $\beta$ for all graphs. Each coordinate is onto because $f_S$ is onto on both derived subgroups.

**Lemma 4.1.** *The actual triangle classes $b_T$ generate $B_\Gamma$ over $R_\Gamma$. Their only nonzero detection coordinates are their own triangle coordinate and the four-cliques containing $T$. For $Q = \{q_1 < \cdots < q_4\}$,*

$$\varepsilon_Q \beta_Q(b_{Q\setminus\{q_r\}}) = (-1)^r. \tag{10}$$

*Proof.* The epimorphism $P_V \to G_\Gamma$ induces an epimorphism on Alexander invariants, and Proposition 3.1 gives its generators. A nonclique triple vanishes if one of the two commutator entries is killed. If only its third edge is absent, the three-strand relation after killing that generator makes the two surviving entries commute. Thus only actual triangles remain.

For a clique $S$ not containing $T$, at least one commutator entry is killed by $f_S$, so its coordinate is zero. The own triangle coordinate is 1 by normalization. For the triangles $123, 234, 134, 124$ of an ordered four-clique, the displayed commutator identities give coordinates $1, -1, Y, -XY$. Their augmentations are $1, -1, 1, -1$, respectively, which is (10). □

*Proof of Theorem 1.1.* Injectivity has been established. Equations (10) and (2), together with generation, show $\lambda\beta = 0$. We prove the stronger inclusion $I_\Gamma D_\Gamma \subseteq \operatorname{im}\beta$.

Fix $Q \in \mathcal{C}_4$. Lemma 3.3 implies that any vector supported only in its essential coordinate, with entry in the augmentation ideal of $R_Q$, has a preimage $b \in B(P_Q)$ whose four local triangle coordinates are zero. Use the standard generator section $P_Q \to G_\Gamma$ [1, Proposition 4.7] to lift $b$. For another target clique $S$, the composite with $f_S$ kills all generators whose endpoints are not both in $A = Q \cap S$; consequently it factors, as an actual group homomorphism, through deletion $P_Q \to P_A$. If $|A| \le 2$, its Alexander image is zero. If $|A| = 3$, the local triangle coordinate of $b$ is zero, and $B(P_A) \to B(F_2)$ is an isomorphism ($P_3 = \mathbb{Z} \times F_2$). Thus every other coordinate vanishes. This proves that pure essential augmentation vectors lift globally, including when four-cliques overlap in triangles.

For a triangle $T$, let $\eta \in I_\Gamma R_T$, and choose $f \in I_\Gamma$ mapping to $\eta$. The vector $\beta(fb_T)$ has only its $T$ triangle coordinate and essential coordinates on four-cliques containing $T$. Those essential entries all have zero augmentation. Subtract the pure essential lifts just constructed to obtain the pure triangle vector $\eta$. Therefore $I_\Gamma D_\Gamma \subseteq \operatorname{im}\beta$.

Modulo $I_\Gamma D_\Gamma$, the columns $\beta(b_T)$ have triangle block the identity and essential block the signed incidence matrix in (10). Their quotient is freely generated by the $c_4$ essential constants, and its

quotient map is exactly $\lambda$. Since $I_\Gamma D_\Gamma \subseteq \operatorname{im}\beta$, this computes the actual cokernel, not just its augmentation fiber. Surjectivity of $\lambda$ also follows by taking a single essential coordinate equal to 1.

For strictness, let $b \in B_\Gamma \cap I_\Gamma D_\Gamma$ and write $b = \sum_T f_T b_T$. Its $T$ coordinate has augmentation $\varepsilon(f_T)$, because no other generator contributes there. Hence all $f_T \in I_\Gamma$, and $b \in I_\Gamma B_\Gamma$. The reverse inclusion is immediate. Together with $I_\Gamma D_\Gamma \subseteq B_\Gamma$, this gives $I_\Gamma B_\Gamma = I_\Gamma D_\Gamma$; multiplication by $I_\Gamma^{r-1}$ proves (3). Finally $\mathbb{Z}^{c_4}$ is free over $\mathbb{Z}$, so $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{c_4}, k) = 0$ for any coefficient ring $k$. Tensoring (1) over $\mathbb{Z}$ therefore preserves injectivity as well as the other exactness assertions. □

## 5. Metabelian separation and integral Chen groups

For a group $H$, write $\gamma_1 H = H$ and $\gamma_{r+1}H = [H, \gamma_r H]$. Its $k$-th Chen group and its rank are, respectively,

$$\gamma_k(H/H'')/\gamma_{k+1}(H/H''), \qquad \theta_k(H).$$

A group is residually torsion-free nilpotent if every nonidentity element has nonidentity image in a torsion-free nilpotent quotient.

**Corollary 5.1.** *Every $G_\Gamma/G_\Gamma''$ is residually torsion-free nilpotent. All its Chen groups are free abelian, and*

$$\theta_1 = e, \qquad \theta_2 = c_3, \qquad \theta_k = (k-1)(c_3 + c_4) \quad (k \geq 3).$$

*Moreover the natural maps give an injection*

$$G_\Gamma/G_\Gamma'' \longrightarrow \mathbb{Z}^e \times (F_2/F_2'')^{c_3+c_4}.$$

*Proof.* Combine abelianization with all $f_S$ modulo their second derived subgroups. An element in the kernel has zero abelianization, hence lies in $B_\Gamma$; Theorem 1.1 then detects it.

For $L = F_2/F_2''$, its derived subgroup is the Laurent module $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$, as computed above. With $J = (X-1, Y-1)$, $\gamma_{r+2}L = J^r L'$, by the conjugation-commutator identity. Taylor substitution $X = 1+U, Y = 1+V$ embeds that Laurent ring into $\mathbb{Z}[[U,V]]$, so $\bigcap_r J^r L' = 0$. Moreover

$$\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/J^r \simeq \mathbb{Z}[U,V]/(U,V)^r$$

is free abelian. Each quotient $L/\gamma_c L$ is an extension of its free abelian abelianization by this torsion-free derived quotient, and is therefore torsion-free and nilpotent. These quotients separate $L$. Finite products and subgroups inherit this residual property; the displayed embedding proves it for $G_\Gamma/G_\Gamma''$.

In general $\gamma_{q+2}(G_\Gamma/G_\Gamma'') = I_\Gamma^q B_\Gamma$. The proof of Theorem 1.1 gives $B_\Gamma/I_\Gamma B_\Gamma = \mathbb{Z}^{c_3}$. For $q \geq 1$, strictness identifies the next quotient with $I_\Gamma^q D_\Gamma / I_\Gamma^{q+1} D_\Gamma$. Each two-variable Laurent summand has degree-$q$ augmentation quotient free of rank $q+1$. This proves both freeness and the formulas, without inferring integrality from rational ranks. □

In particular, the $K_4$-free question of [8, Remark 8.7] has an affirmative answer, strengthened to residual torsion-free nilpotence. When $c_4 = 0$, (1) is an actual direct-sum decomposition; when there are no triangles it gives $B_\Gamma = 0$.

## 6. Rank-one homology in every characteristic

Let $k$ be algebraically closed of any characteristic, and let $\chi : \mathbb{Z}^E \to k^*$ be a character. Write $k_\chi$ for its rank-one local system on $M_\Gamma$. For $S \in \mathcal{C}_3 \cup \mathcal{C}_4$, set

$$\mathbb{T}_S(k) = \{\chi : J_S(\chi) = 0\}.$$

Each is the two-dimensional connected subtorus defined by the equations already displayed. Notice that the four-clique equation is the product on a triangle equal to 1. Replacing it by the product of all six edges would introduce an extra component in characteristic different from 2.

**Lemma 6.1.** *Distinct $\mathbb{T}_S(k)$ meet only at the identity character. For $\chi \neq 1$,*

$$\dim_k H_1(M_\Gamma; k_\chi) = \begin{cases} 1 & \chi \in \mathbb{T}_S(k) \text{ for some } S, \\ 0 & \text{otherwise.} \end{cases}$$

*At the identity the dimension is $e$.*

*Proof.* For different triangles, at most one edge is common. All other coordinates are forced to 1, and either triangle-product equation forces the remaining coordinate to 1. If one clique is a four-clique $Q$, choose a vertex in $Q \setminus S'$. Its three incident $Q$ edges have coordinate 1 on $\mathbb{T}_{S'}$. They represent the three opposite-edge pairs of $Q$, so all six $Q$ coordinates are 1. This argument uses no division by an integer and works in every characteristic.

Tensor (1) over $\mathbb{Z}$ with $k$, and localize at the maximal ideal of $\chi \neq 1$. Some $t_a - 1$ is invertible, so the augmentation cokernel disappears. Thus the localized Alexander invariant is the direct sum of the localized component rings. At most one summand is nonzero, and its character fiber is one-dimensional.

To identify this fiber with twisted homology, use the extension $1 \to G'_\Gamma \to G_\Gamma \to \mathbb{Z}^E \to 1$. The Koszul complex for $H_*(\mathbb{Z}^E; k_\chi)$ is contractible, since a coordinate $\chi_a - 1$ is an invertible scalar. The low-degree homology sequence therefore identifies

$$H_1(G_\Gamma; k_\chi) \simeq (B_\Gamma \otimes_{\mathbb{Z}} k) \otimes_{k[\mathbb{Z}^E]} k_\chi.$$

Changing left/right conventions inverts the character, which preserves every torus here. Degree-one homology of the space agrees with group homology with these coefficients: the universal cover is simply connected, so its first homology is zero. No asphericity assertion is needed. The identity case follows separately from $H_1(M_\Gamma; \mathbb{Z}) = \mathbb{Z}^E$. □

The order of operations matters: localization is exact, and the augmentation cokernel first vanishes there. We do not claim that tensoring an arbitrary injection directly with a character field preserves it.

## 7. Finite abelian covers

Let $\varphi : \mathbb{Z}^E \twoheadrightarrow A$ be a surjection to a finite abelian group. Let $Y \to M_\Gamma$ be its connected regular cover. Abelianizing (9) gives surjections $\alpha_S : \mathbb{Z}^E \twoheadrightarrow \mathbb{Z}^2$, with all unspecified edge vectors sent to zero. The nonzero lists are

$$\begin{aligned} \alpha_T &: \quad (1,0), (0,1), (-1,-1), \\ \alpha_Q &: \quad (1,0), (0,1), (-1,-1), (-1,-1), (0,1), (1,0), \end{aligned}$$

in the respective edge orders of (9). Set $K_S = \ker \alpha_S$, $A_S = A/\varphi(K_S)$, and denote its quotient map by $\pi_S : A \to A_S$. For a finite group $C$, define $\mathrm{Aug}_k(C) = \ker(k[C] \xrightarrow{\varepsilon} k)$ with its regular action. In $\mathrm{Aug}_k(A_S)$ the $A$-action is pulled back by $\pi_S$.

**Theorem 7.1** (Complete first-homology representation)**.** *If $\mathrm{char}\, k = 0$, or $\mathrm{char}\, k = p \nmid |A|$, then as $k[A]$-modules*

$$H_1(Y; k) \simeq k^e \oplus \bigoplus_{S \in \mathcal{C}_3 \cup \mathcal{C}_4} \mathrm{Aug}_k(A_S). \tag{11}$$

*The first summand has trivial action. In particular*

$$b_1(Y; \mathbb{Q}) = e + \sum_S (|A_S| - 1), \qquad b_1(Y; \mathbb{Q}) \leq e + |A| - 1.$$

*Every torsion prime of $H_1(Y; \mathbb{Z})$ divides $|A|$.*

*Proof.* First assume $k$ algebraically closed and $\operatorname{char} k \nmid |A|$. The regular $k[A]$-module splits into characters. The chain complex of $Y$ is the universal-cover chain complex tensored with this module; its decomposition is the sum of the twisted complexes on $M_\Gamma$. Consequently the multiplicity of a deck character $\eta$ in $H_1(Y;k)$ is $\dim_k H_1(M_\Gamma; k_{\eta\varphi})$, up to simultaneous inversion. This is the standard finite-cover character argument [6, Appendix B], included here at chain level.

The pullback $\eta\varphi$ lies in $\mathbb{T}_S$ precisely when $\eta(\varphi(K_S)) = 1$, that is, when $\eta$ factors through $A_S$. Different such character subgroups meet only at 1 by Lemma 6.1 and surjectivity of $\varphi$. The trivial character has multiplicity $e$. All other characters factoring through a given $A_S$ have multiplicity one, and these are exactly the constituents of $\mathrm{Aug}_k(A_S)$. This proves (11) over the splitting field. Inversion leaves each complete character subgroup unchanged.

For arbitrary $k$ in the statement, $k[A]$ is a product of finite separable field extensions of $k$. A module over this product is determined by its dimension over each field factor, and these dimensions are detected after extension to an algebraic closure. Hence the isomorphism descends to $k$. Disjointness of the nontrivial character sets gives $\sum_S (|A_S| - 1) \le |A| - 1$.

Finally $Y$ has finite CW type, so $H_1(Y;\mathbb{Z})$ is finitely generated. Formula (11) over $\mathbb{Q}$ and over $\mathbb{F}_p$, for $p \nmid |A|$, gives the same dimension. The universal coefficient theorem in degree one has no extra term, since $H_0(Y;\mathbb{Z}) = \mathbb{Z}$. Thus this dimension equality says that no $p$-primary invariant factor occurs in $H_1(Y;\mathbb{Z})$. This proves the torsion-prime assertion. □

7.1. **Computing the quotient groups.** The full integral kernels $K_S$ are generated by the outside edge vectors and, respectively,

$$u_{ij} + u_{ik} + u_{jk},$$

or

$$u_{ij} - u_{k\ell}, \quad u_{ik} - u_{j\ell}, \quad u_{i\ell} - u_{jk}, \quad u_{ij} + u_{ik} + u_{jk}.$$

Indeed, quotienting by these relations leaves a free group of rank two with basis the $ij$ and $ik$ edge classes. Thus Smith normal form of the labels $\varphi(u_a)$ computes the actual finite quotient $A_S$, without passing to a rational span.

**Example 7.2.** For two disjoint triangles and $A = (\mathbb{Z}/p)^2$, label their ordered edges by $(x, 0, -x)$ and $(y, 0, -y)$, where $x, y$ form a basis of $A$. Then $A_T = A/\langle y\rangle$ and $A_{T'} = A/\langle x\rangle$ both have order $p$, and $b_1(Y) = 6 + 2(p-1)$. Their character subgroups are distinct lines and meet only at 1. Thus disjoint character sets do not imply coprime quotient orders for a general finite abelian deck group.

For $a \in A$, let $o_S(a)$ be the order of $\pi_S(a)$. A further explicit consequence is

$$\det(t\,\mathrm{id} - a_* \mid H_1(Y;\mathbb{Q})) = (t-1)^e \prod_S \frac{(t^{o_S(a)} - 1)^{|A_S|/o_S(a)}}{t-1}.$$

The regular action on $\mathbb{Q}[A_S]$ has $|A_S|/o_S(a)$ cycles of length $o_S(a)$; removing its one-dimensional trivial summand proves this formula.

## 8. All positive weights and the Milnor formula

Assume $e \ge 1$. Choose positive integers $m_a$ with $\gcd(m_a : a \in E) = 1$, and set

$$N = \sum_{a\in E} m_a, \qquad f_m(z) = \prod_{ij\in E} (z_i - z_j)^{m_{ij}}, \qquad F_m = f_m^{-1}(1).$$

The geometric monodromy is $h(z) = \exp(2\pi i/N)z$. For $T \in \mathcal{C}_3$ and $Q = \{i<j<k<\ell\} \in \mathcal{C}_4$, put

$$d_T = \gcd(N, \{m_a : a \notin E(T)\}),$$

$$d_Q = \gcd(N, \{m_a : a \notin E(Q)\}, m_{ij} - m_{k\ell}, m_{ik} - m_{j\ell}, m_{i\ell} - m_{jk}, m_{ij} + m_{ik} + m_{jk}). \tag{12}$$

An empty outside list is allowed; in particular $d_T = N$ when it is empty. The gcd is positive, with signs and zero entries interpreted in the usual way. Modulo the opposite-edge differences all four triangle sums in $Q$ agree. Hence $d_Q$ is independent of the ordering of $Q$.

**Theorem 8.1** (Arbitrary-weight graphic monodromy)**.** *The fiber $F_m$ is connected, and as a rational representation of $C_N$,*

$$H_1(F_m;\mathbb{Q}) \simeq \mathbb{Q}^{e-1} \oplus \bigoplus_{T\in\mathcal{C}_3} \mathrm{Aug}_{\mathbb{Q}}(C_{d_T}) \oplus \bigoplus_{Q\in\mathcal{C}_4} \mathrm{Aug}_{\mathbb{Q}}(C_{d_Q}). \tag{13}$$

*The action on each augmentation summand is through $C_N \to C_{d_S}$. In particular,*

$$\Delta_m(t) = (t-1)^{e-1} \prod_{S\in\mathcal{C}_3\cup\mathcal{C}_4} \frac{t^{d_S}-1}{t-1},$$

$$b_1(F_m) = e - 1 + \sum_S (d_S - 1).$$

*Monodromy is trivial over $\mathbb{Q}$ if and only if every $d_S = 1$. Every torsion prime of $H_1(F_m;\mathbb{Z})$ divides $N$.*

*Proof.* Let $U = \mathbb{P}(M_\Gamma)$. The map $F_m \to U$ is a cyclic cover of degree $N$: a projective point has $N$ scalar representatives satisfying $f_m = 1$. Its classifying homomorphism sends an edge meridian to $m_a$ modulo $N$. This is well-defined on $H_1(U;\mathbb{Z}) = \mathbb{Z}^E/\mathbb{Z}(\sum_a u_a)$ because the weights sum to $N$. The gcd hypothesis makes it onto, hence the cover is connected.

Choose a defining linear form of the arrangement. Since it never vanishes on $M_\Gamma$, normalization by that form trivializes $M_\Gamma \simeq U \times \mathbb{C}^*$. Pull back $F_m \to U$ to this product. The resulting connected $C_N$-cover $Y \to M_\Gamma$ is $F_m \times \mathbb{C}^*$, with deck action trivial on the second factor. It is classified by $\varphi(u_a) = m_a \bmod N$. Thus $H_1(Y;\mathbb{Z}) = H_1(F_m;\mathbb{Z}) \oplus \mathbb{Z}$, with trivial action on the last term.

Apply the integral kernel generators from Section 7. For a triangle, the images of the outside weights and the internal sum generate the same subgroup of $C_N$ as the outside weights, because their total is $N$. The quotient is $C_{d_T}$. For a four-clique the listed opposite differences and a triangle sum give precisely $C_{d_Q}$. Theorem 7.1, followed by removal of the trivial $\mathbb{C}^*$ summand, proves (13), the polynomial, and the torsion bound. Each nontrivial cyclic augmentation summand has nonidentity eigenvalues, so the stated triviality criterion follows as well. □

Distinct $d_S$ are pairwise coprime in this cyclic case. Otherwise a nontrivial character of a common prime order would lie in two of the disjoint component character subgroups. Thus all nonidentity eigenvalues in $\Delta_m$ are simple. This conclusion uses cyclicity and is compatible with the noncyclic example above.

**Example 8.2** (Boundary cases and a fixed-graph family)**.** For one edge, primitivity forces $N = 1$, and the formula gives $b_1 = 0$. A single triangle has $d_T = N$ and gives $b_1 = N + 1$. For $K_4$, assign positive weights $a, b, c$ to its three opposite-edge pairs, with $\gcd(a, b, c) = 1$. Then $N = 2(a + b + c)$, all $d_T = 1$, and $d_Q = a + b + c$. Consequently

$$\Delta_m(t) = (t-1)^5 \frac{t^{a+b+c}-1}{t-1}.$$

Taking $(a, b, c) = (1, 1, d-2)$ produces eigenvalues of arbitrarily large order on this fixed arrangement. This is an arbitrary-weight statement; the reduced specialization is classical [5].

## 9. A direct negative answer to the weighted question

**Proposition 9.1.** *The rank-three arrangement with reduced polynomial $xy(x - y)z$ is decomposable, and its Alexander invariant is augmentation-separated over $\mathbb{Z}$ and over $\mathbb{Q}$. Nevertheless for weights*

$(1,1,1,3)$ *the monodromy of* $f = xy(x-y)z^3$ *on* $H_1(F;\mathbb{Q})$ *has characteristic polynomial*

$$(t-1)^3(t^2+t+1).$$

*The integral monodromy is also nontrivial.*

*Proof.* Use $u = y/x$ to identify its complement with $(\mathbb{C}^*)^2 \times (\mathbb{C} \setminus \{0,1\})$. Its group is $\mathbb{Z}^2 \times F_2$. To check decomposability explicitly, define the holonomy Lie algebra by the free Lie ring on $H^1(M;\mathbb{Z})^*$ modulo the image of the dual cup product $H^2(M;\mathbb{Z})^* \to \bigwedge^2 H^1(M;\mathbb{Z})^*$. The product description gives $\mathfrak{h}(M) = \mathbb{Z}^2 \oplus \mathrm{FreeLie}_{\mathbb{Z}}(\mathbb{Z}^2)$, with the first summand central. Its only rank-two pencil of multiplicity at least three is the triple $x, y, x-y$. Projection to that pencil identifies the derived holonomy algebra with the derived algebra of its rank-two free factor; all double pencils have zero derived algebra. Thus the natural comparison with the sum of local derived holonomy algebras is an integral isomorphism, which is the decomposability criterion used in the question. Its Alexander invariant is the rank-one Laurent module of $F_2$, with the two central directions acting trivially. The Taylor argument in the proof of Corollary 5.1 proves separation.

On the fiber, set $v = xz$. There is an explicit algebraic isomorphism

$$F \simeq \mathbb{C}^* \times C_0, \qquad C_0 = \{(u,v) : u \notin \{0,1\},\ v^3u(1-u) = 1\},$$

with inverse $y = ux$, $z = v/x$. The curve $C_0$ is a connected threefold cyclic cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, fully ramified at the three punctures in its smooth compactification. Riemann–Hurwitz gives genus one, and deleting those three points gives $b_1(C_0) = 4$.

For $\zeta = \exp(2\pi i/6)$, monodromy acts by $(x,u,v) \mapsto (\zeta x, u, \zeta^2 v)$. It is trivial on the first $\mathbb{C}^*$ homology. On the curve, the two puncture classes are fixed, while the compact genus-one part has the two primitive cube-root eigenvalues. One can see the latter directly from the cyclic-cover character complex: each nontrivial character on the three-punctured sphere has zero $H_0$, and Euler characteristic $-1$, hence one-dimensional $H_1$. The invariant part has dimension two. The displayed polynomial follows. The integral homology of the punctured curve, and hence of $F$, is free; nontrivial rational action therefore implies nontrivial integral action. □

The proposition answers both literal arbitrary-weight questions in the version [8] specified in the bibliography. It also pinpoints the arithmetic issue in the proof of its Theorem 12.1: a proper coordinate torus can meet a weighted one-parameter torus in nonidentity roots of unity. In this example the outside equation is $\lambda^3 = 1$, not $\lambda = 1$. Formula (12) records the correct intersection. For all weights equal to one, an outside coordinate does force $\lambda = 1$; the example does not contradict the reduced-fiber assertion.

## 10. Scope of the conclusions

The structural theorem concerns all finite simple ordinary graphic arrangements, including disconnected graphs and isolated vertices. For $E = \varnothing$, $G_\Gamma$ and $B_\Gamma$ are trivial, and the finite-cover statement is vacuous except for the trivial cover. A positive-degree Milnor fibration requires $e \geq 1$, as imposed above.

Theorem 7.1 gives an isomorphism of representations in good characteristic. We have not yet specified a preferred integral splitting or determined the torsion at primes dividing the covering degree. Those primes require nonsemisimple homology information. We have not yet extended the separation argument in this note to arbitrary decomposable arrangements.

The sources supply the classical pure-braid presentation, free-group maps, and finite-cover method. References to Question 12.3 concern arXiv:2404.04784v2.

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