INTEGRAL MODELS OF COMPACT ARRANGEMENTS

THE GORINOV–ZAKHAROV INTEGER-MODEL QUESTION AND THE DELUCCHI–PAGARIA CELLULAR-MODEL QUESTION

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Abstract. We answer two questions about models of complements of compact arrangements. For a commutative coefficient ring, the Mayer–Vietoris complex of the intersection cohomology groups computes relative cohomology for every finite compact affine subtorus arrangement if and only if the ring is a $\mathbb{Q}$-algebra. Necessity is detected by two embedded circles in a two-torus; their integral relative cohomology is free, whereas the proposed complex has an extra cyclic summand of any prescribed order. The integral spectral sequence for this example collapses at E_2: a nontrivial extension accounts for the discrepancy. This gives a negative answer to the question following Theorem 3 of Gorinov–Zakharov. We then construct a finite simplicial model, embedded as a strong deformation retract of the complement, for every arrangement given by integer torus equations with arbitrary phases. Faithful lattice blocks realize all endomorphisms of an elliptic curve, including complex multiplication, so the construction answers the general cellular-model problem in Delucchi–Pagaria's Remark 2.3. The model retains integer boundaries, cup products, local coefficients, and finite covers, including arrangements with disconnected intersections.

1. The two questions and the information retained by a model

Write $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. A compact affine subtorus in this note is a translate of a closed subgroup of $\mathbb{T}^N$; the subgroup may have finitely many connected components. For a finite arrangement $\mathcal{A} = \{T_1, \dots, T_s\}$, put

$$A = \bigcup_{i=1}^{s} T_i, \qquad M(\mathcal{A}) = \mathbb{T}^N \setminus A, \qquad T_I = \bigcap_{i\in I} T_i, \quad T_\varnothing = \mathbb{T}^N.$$

Empty intersections contribute zero cohomology. All coefficient rings are commutative and unital, and all module isomorphisms respect the coefficient ring.

Gorinov–Zakharov prove that, for a $\mathbb{Q}$-algebra R, the complex

$$\mathrm{MV}\{H^*(T_I; R)\}$$

computes $H^*(\mathbb{T}^N, A; R)$ [2, Theorem 3]. The cohomology groups are regarded as complexes with zero internal differential; the only differential is the alternating sum of actual restriction maps. They ask whether the same conclusion holds over $\mathbb{Z}$, both immediately after Theorem 3 and in their discussion of the size of the models in [2, Section 2.2]. Their integral binomial models retain additional diagram data; the question is whether the smaller cohomology complex suffices.

Delucchi–Pagaria construct cellular models for a specified class of integer elliptic arrangements. Their Remark 2.3 asks for models in the general setting of equations

$$\sum_{j=1}^{d} \alpha_{e,j} z_j = c_e, \qquad \alpha_{e,j} \in \mathrm{End}(E), \quad c_e \in E, \tag{1.1}$$

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where E is an elliptic curve and the phases are arbitrary. Remark 3.12 suggests studying the arrangement in the universal cover [1]. We follow that geometric direction, using a fine periodic subdivision to obtain a regular finite quotient and a direct deformation retraction. The subdivision and barycentric retraction themselves are classical; their compatibility with all the equations and phases is proved below.

Theorem 1.1 (Sharp coefficient-ring classification)**.** *For a commutative unital ring R, the following are equivalent:*

1. *For every finite compact affine subtorus arrangement, the graded cohomology of* $\mathrm{MV}\{H^*(T_I;R)\}$ *is isomorphic to $H^*(\mathbb{T}^N, A; R)$.*
2. *Every positive integer is invertible in R; equivalently, R is a $\mathbb{Q}$-algebra.*

Necessity already holds for arrangements of two connected circles in $\mathbb{T}^2$ passing through the identity.

Theorem 1.2 (A phase-preserving finite cellular model)**.** *For each label e in a finite set, let*

$$A_e \in \mathrm{Mat}_{r_e,N}(\mathbb{Z}), \qquad b_e \in \mathbb{T}^{r_e}, \qquad H_e = \{x \in \mathbb{T}^N : A_e x = b_e\}.$$

There is an explicitly defined finite face poset with a subposet of good faces whose order complex $S(A,b)$ embeds in $\mathbb{T}^N \setminus \bigcup_e H_e$ as a strong deformation retract. The construction permits arbitrary phases, dependent or nonprimitive rows, disconnected kernels, repeated labels, and empty deleted sets. In particular it gives a finite cellular model for every arrangement (1.1), and for every finite arrangement of translates of abelian subvarieties in a compact abelian variety.

The first theorem determines when intersection cohomology is sufficient for this additive computation. The second constructs the homotopy type from the equations and phases, retaining the boundary maps responsible for integral extensions.

2. Two circles and the exact coefficient discrepancy

2.1. **The complement and its relative cells.** Fix an integer $m \geq 2$. In $T = \mathbb{T}^2$ with coordinates (x, y), consider

$$C_0 = \{(t, 0) : t \in \mathbb{T}\}, \qquad C_m = \{(t, mt) : t \in \mathbb{T}\}, \qquad A_m = C_0 \cup C_m.$$

The first coordinate in each parametrization is t, so both circles are embedded and connected. Their intersection consists of

$$P_j = (j/m, 0), \qquad 0 \leq j < m.$$

The torus homomorphism

$$F : T \longrightarrow \mathbb{T}^2, \qquad F(x, y) = (y, mx - y) \tag{2.1}$$

has determinant $-m$ and is a covering of degree m. The inverse image of the union of the two coordinate circles is exactly A_m. The complement of those coordinate circles is an open square, hence simply connected. Its inverse image under the covering is therefore the disjoint union of m open squares.

The two circles and these squares form a finite CW decomposition of T: A_m is its one-skeleton, with m vertices and $2m$ edges, and there are m two-cells. Consequently the relative cellular cochain complex for (T, A_m) has R^m in degree two and zero in every other degree, for any ring R. In particular

$$H^k(T, A_m; R) = \begin{cases} R^m, & k = 2, \\ 0, & k \neq 2. \end{cases} \tag{2.2}$$

The relative cellular calculation is valid over every R.

2.2. **The cohomology-diagram complex.** We use the degree convention in [2]:

$$C^{p,q} = \bigoplus_{|I|=p} H^q(T_I; R), \qquad \text{total degree} = p+q.$$

For $I = \{i_1 < \cdots < i_{p+1}\}$ the horizontal differential is

$$(\delta c)_I = \sum_{j=1}^{p+1} (-1)^{j+1} c_{I\setminus\{i_j\}}|_{T_I}.$$

The internal differential is zero. With two circles there are three nonzero horizontal rows.

In degree $q = 0$ the row is

$$R \longrightarrow R^2 \longrightarrow R^m, \qquad c \longmapsto (c,c), \quad (a,b) \longmapsto (b-a)(1,\dots,1).$$

The first map is a split injection; the kernel of the second is its image. The constant vectors form a split copy of R in R^m, so the only row cohomology is R^{m-1} at $p = 2$.

In degree $q = 1$ the row is $R^2 \to R^2 \to 0$. The restrictions of dx, dy to the two circles give the matrix

$$B_m = \begin{pmatrix} 1 & 0 \\ 1 & m \end{pmatrix}. \tag{2.3}$$

Subtracting its first row from its second transforms it to $\operatorname{diag}(1,m)$. These are invertible integer operations over every R. Its kernel is $\operatorname{Ann}_R(m)$ and its cokernel is R/mR. Finally, the $q = 2$ row is $R \to 0 \to 0$. Combining total degrees gives

$$H^k \operatorname{MV}\{H^*(T_I; R)\} = \begin{cases} \operatorname{Ann}_R(m), & k = 1, \\ R^m \oplus R/mR, & k = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

For $R = \mathbb{Z}$ the model has an extra $\mathbb{Z}/m$ summand; for $R = \mathbb{F}_p$ and $p \mid m$ it has an extra one-dimensional group in both degrees one and two. The discrepancy can thus be detected before considering any ring structure.

Proof of Theorem 1.1. Sufficiency is [2, Theorem 3]; all its hypotheses are precisely those in the statement for a $\mathbb{Q}$-algebra. Suppose conversely that the small model works for every arrangement. Equations (2.2) and (2.4) imply

$$R^m \oplus R/mR \cong R^m$$

for every $m \geq 2$. If m were not invertible, the proper ideal mR would lie in a maximal ideal $\mathfrak{n}$. Tensoring this isomorphism with the residue field $k = R/\mathfrak{n}$ would give $k^{m+1} \cong k^m$, which is impossible. Tensoring an isomorphism does not require exactness of the tensor functor. Hence every $m \geq 2$ is invertible, and the universal localization $\mathbb{Z} \to \mathbb{Q}$ factors the structure map to R. This is exactly a $\mathbb{Q}$-algebra structure. The zero ring, if allowed among unital rings, has its unique such structure and satisfies both conditions. □

3. An extension in the integral Mayer–Vietoris filtration

The integral example identifies the missing information explicitly. Index all edges and faces modulo m. Let h_j be the horizontal edge from P_j to P_{j+1}, and let v_j be the edge of the sloping circle making one positive y-turn from P_j to P_{j+1}. Orient the jth face so that

$$\partial f_j = h_j + v_{j+1} - h_{j+1} - v_j. \tag{3.1}$$

For example, its lifted sides are $y = 0$, $y = 1$, and $mx - y = j, j+1$, with the resulting orientations. This verifies (3.1) in the torus.

Put

$$L_0 = \{c \in \mathbb{Z}^m : \textstyle\sum_j c_j = 0\}, \qquad (Du)_j = u_j - u_{j+1}.$$

An edge cochain on A_m with values (a_j, b_j) has connecting image in $H^2(T, A_m; \mathbb{Z}) = \mathbb{Z}^m$ equal to $D(a-b)$, by evaluating (3.1). All cochains on the graph are cocycles. A vertex coboundary changes a and b by the same difference vector, so this formula is well defined on graph cohomology. Its image is L_0: the image of D consists exactly of the integer vectors of sum zero, as follows either by successive summation or by solving the difference equations.

The restrictions to the two individual circles have periods $\sum_j a_j$ and $\sum_j b_j$. Their simultaneous kernel therefore has connecting image exactly DL_0. Indeed, if both periods vanish then $a - b \in L_0$, and conversely every $u \in L_0$ is obtained with $(a, b) = (u, 0)$.

For completeness, the filtration of relative cohomology supplied by the two-set Mayer–Vietoris construction is

$$H = \mathbb{Z}^m \supset F^1H = L_0 \supset F^2H = DL_0 \supset F^3H = 0. \tag{3.2}$$

The term F^1H is the image of $H^1(A_m)$ under the relative connecting map. The second is the image of the kernel of $H^1(A_m) \to H^1(C_0) \oplus H^1(C_m)$; the union Mayer–Vietoris sequence identifies that kernel with the image from the intersection term. These descriptions are also obtained by the filtration of the iterated relative mapping cone, so they give the spectral-sequence filtration. The map $H^2(T, A_m) \to H^2(T)$ is the sum of the face coefficients, which confirms $H/F^1H \cong \mathbb{Z}$.

Proposition 3.1. *The sequence*

$$0 \longrightarrow DL_0 \longrightarrow L_0 \longrightarrow \mathbb{Z}/m \longrightarrow 0$$

is exact and nonsplit. Thus the integer Mayer–Vietoris spectral sequence collapses at E_2 but its graded pieces are not a direct sum decomposition of the total relative cohomology.

Proof. Define

$$\theta : L_0 \longrightarrow \mathbb{Z}/m, \qquad \theta(c) = \sum_{j=0}^{m-1} jc_j \pmod m.$$

A reindexing shows $\theta(Du) = \sum_j u_j \pmod m$, so $DL_0 \subseteq \ker\theta$. Also $\theta(e_1 - e_0) = 1$, so θ is onto. If $c \in \ker\theta$, first solve $Du = c$ over $\mathbb{Z}$, using $\sum_j c_j = 0$. The identity just proved says $\sum_j u_j$ is divisible by m. Adding a suitable constant vector makes this sum zero without changing Du, so $c \in DL_0$. This proves exactness. The middle group is torsion-free and the quotient is nonzero torsion, which precludes a splitting. The map D on L_0 is injective, since a constant vector of sum zero is zero; hence $DL_0 \cong \mathbb{Z}^{m-1}$.

The only nonzero E_2 terms are

$$E_2^{0,2} = \mathbb{Z}, \qquad E_2^{1,1} = \mathbb{Z}/m, \qquad E_2^{2,0} = \mathbb{Z}^{m-1}.$$

Every differential d_r, $r \geq 2$, has zero source or target, so $E_2 = E_\infty$. Their total direct sum is $\mathbb{Z}^m \oplus \mathbb{Z}/m$, whereas (3.2) has total group $\mathbb{Z}^m$. The nonsplit sequence is precisely the extension that joins the two positive-filtration pieces. □

Remark 3.2. The integral collapse statement requires a separate analysis after base change to $\mathbb{F}_p$ when $p \mid m$: an additional $E_2^{0,1}$ term may then support d_2. The direct calculation (2.4) already proves failure of the small complex over those fields.

4. A finite set of primes for a fixed arrangement

For a fixed arrangement, only finitely many primes obstruct the additive computation. The relevant primes are read from the restriction maps on all intersection components.

Proposition 4.1 (Localization of the additive model)**.** *For a fixed compact affine subtorus arrangement, take the integral horizontal matrices in*

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(T_I; \mathbb{Z}).$$

Let $\mathcal{P}$ contain the primes dividing their nonzero Smith invariant factors, and set $a = \prod_{p\in\mathcal{P}} p$, with empty product one. If every prime in $\mathcal{P}$ is a unit in a commutative ring R, then

$$H^*\,\mathrm{MV}\{H^*(T_I;R)\} \cong H^*(\mathbb{T}^N, A; R)$$

as graded R-modules. The isomorphism is generally noncanonical. For the two-circle arrangement this condition is sharp: its small model works exactly when m is a unit in R.

Proof. Every intersection is a finite union of tori, so each E_1 term is a finite free abelian group. For adjacent maps $U \xrightarrow{A} V \xrightarrow{B} W$ with $BA = 0$, there is an exact sequence

$$0 \longrightarrow \ker B/\operatorname{im} A \longrightarrow \operatorname{coker} A \longrightarrow \operatorname{im} B \longrightarrow 0.$$

The last group is free, so the sequence splits; in particular the torsion primes of the cohomology are among the Smith primes of A. Over the PID $R_0 = \mathbb{Z}[1/a]$ all E_2 terms are consequently finite free. The rational Mayer–Vietoris spectral sequence degenerates at E_2, as proved in [2, Theorem 3 and Section 3]. Inductively, a higher differential over R_0 becomes zero over $\mathbb{Q}$ and has torsion-free target; it is therefore zero. Each page remains the free E_2 page. The finite filtration on total cohomology has free quotients, so its extensions split over R_0, giving the asserted additive isomorphism over that ring.

To allow arbitrary, possibly nonflat R, choose a finite cellular decomposition of the torus in which all deleted subtori are subcomplexes; one is constructed in Sections 5–6 below. The relative cellular complex and the small Mayer–Vietoris complex over R_0 are both finite free complexes, with finite free cohomology just proved isomorphic. Any such complex over a PID is chain homotopy equivalent to its cohomology with zero differential: boundaries and cycles are free, and the sequences with free quotient split degree by degree. Choose these splittings and the cohomology isomorphism. They give a noncanonical chain homotopy equivalence between the two complexes. Tensoring a chain homotopy equivalence with any R_0-algebra remains a chain homotopy equivalence. Here R is an R_0-algebra by the unit hypothesis. Since all the individual torus cohomology groups are free, change of coefficients gives exactly the R-cohomology diagram; the relative cellular complex also changes coefficients directly. This proves the statement for every R_0-algebra R. For two circles, the only nonunit Smith factor is m; equations (2.2)–(2.4) and the maximal ideal argument in Theorem 1.1 prove sharpness. □

The localization isomorphism is additive and depends on the chosen splittings. Cup products will be computed directly from the finite model in Section 7.

5. Faithful equations for compact and elliptic arrangements

5.1. **Closed subgroups and integer equations.** The following realization also explains why finite component groups must be retained.

Lemma 5.1. *Every translate of a closed subgroup of $\mathbb{T}^N$ is the fiber of an integer torus homomorphism $\mathbb{T}^N \to \mathbb{T}^r$ over some phase.*

Proof. Let K be the subgroup. Its identity component K^0 is a connected subtorus. Its Lie algebra is rational with respect to $\mathbb{Z}^N$, because its intersection with the ambient lattice is a full lattice in that Lie algebra. The annihilator of this rational space is a primitive integer sublattice. An integer basis, completed by Smith normal form, yields a quotient $\mathbb{T}^N \to \mathbb{T}^r$ whose kernel is exactly K^0. The image $F = K/K^0$ is finite. Choose an integer m killing F. The characters of $(\mathbb{Z}/m)^r$ separate its points and those of any quotient of it: this follows from the decomposition of a finite abelian group into cyclic groups and the characters of each cyclic factor. Thus F is exactly the common zero set in $\mathbb{T}^r$ of the integer characters annihilating it. Points outside the m-torsion are already separated by the characters mx_j. Lifting the remaining characters to integer rows gives a finite integer matrix whose kernel upstairs is K. Finally, translating by t replaces the zero phase by At. □

Allowing zero or dependent rows is convenient. A zero row with phase zero is a vacuous condition; a zero row with nonzero phase makes the fiber empty. An empty row list defines the whole torus. These conventions will remain part of each label.

5.2. **Endomorphisms of an elliptic curve.** Let

$$E = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2),$$

where ω_1, ω_2 are real-linearly independent. Each $\alpha \in \mathrm{End}(E)$ lifts to multiplication by a complex number preserving the lattice, and there is a unique integer matrix B_α satisfying

$$(\alpha\omega_1, \alpha\omega_2) = (\omega_1, \omega_2)B_\alpha.$$

This representation respects addition and composition. It is faithful: $B_\alpha = 0$ implies $\alpha\omega_1 = 0$, hence $\alpha = 0$. In the real coordinates $z = x\omega_1 + y\omega_2$, the map $\alpha : E \to E$ is $(x, y)^{\mathsf{T}} \mapsto B_\alpha(x, y)^{\mathsf{T}}$ modulo $\mathbb{Z}^2$. Therefore (1.1) is precisely

$$[B_{\alpha_{e,1}} \ \cdots \ B_{\alpha_{e,d}}]x = b_e \quad \text{in } \mathbb{T}^2. \tag{5.1}$$

Both coordinates of c_e enter the block equation. The construction uses the lattice action of $\mathrm{End}(E)$ and applies to every such ring.

For example, on $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, multiplication by i has matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The equation $z_1 + iz_2 = u + iv$ becomes $x_1 - y_2 = u$ and $y_1 + x_2 = v$ modulo $\mathbb{Z}$. These real equations describe the same kernel and its components.

More generally a compact abelian variety has underlying real torus $\mathbb{T}^{2g}$. Its abelian subvarieties are closed connected subgroups, so Lemma 5.1 realizes their translates simultaneously in real torus coordinates. The cellular construction applies even to underlying compact complex tori that are not projective.

6. A small periodic subdivision and its finite regular quotient

Fix the data in Theorem 1.2. Choose representatives $\beta_e \in [0, 1)^{r_e}$ for the phases. For every nonzero row λ of every A_e, with its corresponding phase coordinate β, include all real hyperplanes

$$\lambda(x) = \beta + k, \qquad k \in \mathbb{Z}. \tag{6.1}$$

Also include the auxiliary grid

$$3x_j = k, \qquad k \in \mathbb{Z}, \quad 1 \le j \le N. \tag{6.2}$$

The grid subdivides the ambient space while preserving the original deleted set. Zero-row conditions are recorded as the vacuous or impossible conditions specified above.

Lemma 6.1. *These hyperplanes give a locally finite $\mathbb{Z}^N$-periodic polyhedral complex $\widetilde{K}$. Every closed face is a bounded convex polytope, and each of its coordinate projections has length at most $1/3$. Its quotient $K = \widetilde{K}/\mathbb{Z}^N$ is a finite regular polyhedral CW complex, with the intersection of two closed cells either empty or one common face.*

Proof. Integer translation changes $\lambda(x)$ by an integer, and also preserves the grid; hence the collection is periodic. On any compact subset each row is bounded, so only finitely many of its integer levels occur. Since there are finitely many rows, local finiteness follows. The usual face decomposition of a finite hyperplane arrangement on each bounded grid box agrees on overlaps, giving the polyhedral complex. Each face closure lies in one closed grid box and is a convex polytope.

The quotient map is injective on each closed face: two points there that differ by an integer vector differ in each coordinate by at most $1/3$, forcing the vector to be zero. Moreover, for two fixed lifted closed faces P, Q, a translate $Q + v$ meeting P must have each integer v_j in the difference interval $P_j - Q_j$. That interval has length at most $2/3 < 1$, so it contains at most one integer. At most one translate can thus meet P. Its intersection is one common face in the lifted complex. Each quotient

cell has an embedded closed-ball closure, and its boundary is the union of the images of its proper faces. The uniqueness of the meeting translate identifies intersections with common faces. This proves the stated regularity.

Every face orbit has a closure meeting $[0,1]^N$. Such a face is contained in $[-1,2]^N$ by the grid bound. Only finitely many hyperplanes meet this larger cube, so there are only finitely many such faces and hence finitely many face orbits. The quotient is therefore a finite regular complex on $\mathbb{T}^N$. □

Let $\mathcal{P}$ be the poset of nonempty faces of K, ordered by inclusion of closures. For a face F, define

$$e \in \operatorname{supp}(F) \quad \Longleftrightarrow \quad \text{all rows of } A_e \text{ satisfy their phase equations on } F. \tag{6.3}$$

The definition can be made on any lift of F and is unchanged by translation. On an open face, a nonzero row either equals one of its phase levels everywhere or lies strictly between levels. Thus the truth of every condition is constant. Zero rows use their separately recorded truth values.

Lemma 6.2. *Let $\mathcal{P}_{\mathrm{bad}} = \{F : \operatorname{supp}(F) \neq \varnothing\}$ and $\mathcal{P}_{\mathrm{good}} = \mathcal{P} \setminus \mathcal{P}_{\mathrm{bad}}$. The bad faces are an order ideal. Their union is exactly $\bigcup_e H_e$, as a closed subcomplex of K.*

Proof. The constant row states show that a face is contained in H_e exactly when $e \in \operatorname{supp}(F)$. Since H_e is closed, every boundary face of such a face is contained in the same H_e. This proves the order-ideal assertion. Every point lies in a unique open face, where all row states are fixed; hence no partial open face is deleted. The union of the bad cells is therefore exactly the stated union of fibers. □

The construction is finite for every choice of real phases. For rational or torsion phases, the defining inequalities and all face incidences can be obtained by exact rational polyhedral operations.

7. Deleting bad vertices by a strong deformation retraction

Write $\Delta(\mathcal{P})$ for the order complex: its vertices are faces, and its simplices are strict chains of faces. Choose a barycenter in each lifted convex face, compatibly with integer translation. Barycentric subdivision of every face closure gives a triangulation; the subdivisions agree on their common faces. For a chain of quotient faces, lift its largest face. The uniqueness of a meeting translate in Lemma 6.1 gives unique nested lifts of all its subfaces. Thus the barycentric simplices are unambiguously defined in the quotient, and their face identifications are precisely those of $\Delta(\mathcal{P})$. This gives a PL homeomorphism

$$h : |\Delta(\mathcal{P})| \longrightarrow \mathbb{T}^N.$$

By Lemma 6.2, $|\Delta(\mathcal{P}_{\mathrm{bad}})|$ maps exactly to the deleted set. The subcomplex on the bad vertices is induced: a chain whose vertices are all bad is a simplex of this order complex.

For z in its complement, let $\lambda_F(z)$ be its barycentric coordinates on a simplex containing it; omitted coordinates are zero. Define

$$w(z) = \sum_{F \in \mathcal{P}_{\mathrm{good}}} \lambda_F(z) > 0.$$

For $0 \le t \le 1$ change the coordinates by

$$\lambda_F(z,t) = \begin{cases} (1-t)\lambda_F(z), & F \in \mathcal{P}_{\mathrm{bad}}, \\ \big((1-t) + t/w(z)\big)\lambda_F(z), & F \in \mathcal{P}_{\mathrm{good}}. \end{cases} \tag{7.1}$$

They remain nonnegative and sum to one. The total good weight is $(1-t)w(z) + t > 0$, so the path remains in the complement. It stays in the same simplex. On a common face all omitted coordinates are zero and the formulas agree; w is continuous and positive on the complement, so the resulting homotopy is continuous there. At $t = 1$ only good coordinates remain, giving a simplex of $\Delta(\mathcal{P}_{\mathrm{good}})$. If all coordinates were good initially, then $w = 1$ and the point is fixed throughout.

Proof of Theorem 1.2. Take

$$S(A,b) = \Delta(\mathcal{P}_{\text{good}}).$$

The homeomorphism h and (7.1) give the required embedded strong deformation retract. The construction and all previous lemmas allow every stated type of row and phase. If a label defines the whole torus, every face is bad and both the complement and the model are empty. If all labels define empty sets, every face is good and the model triangulates the whole torus. For $N = 0$ the space is a point, and the same empty/full convention proves the assertion directly. The elliptic and abelian cases follow from the faithful realizations in Section 5. □

8. Integer boundaries, cup products, and local coefficients

The model gives an explicit integer chain complex. Write S for its embedded realization $|S(A,b)|$. Choose the chain order to orient each simplex. The integer complex is

(8.1)

$$C_k = \mathbb{Z}\{F_0 < \cdots < F_k : F_i \in \mathcal{P}_{\text{good}}\}, \qquad \partial[F_0 < \cdots < F_k] = \sum_{j=0}^{k} (-1)^j [F_0 < \cdots < \widehat{F}_j < \cdots < F_k].$$

Here the boundary formula is used for $k \geq 1$, and $\partial_0 = 0$. Each double deletion occurs twice with opposite signs, proving $\partial^2 = 0$. The strong deformation retract identifies its homology with the integral homology of the complement. For any ring R, $\mathrm{Hom}_{\mathbb{Z}}(C_*, R)$ computes its R-cohomology.

On these simplicial cochains the ordinary cup product is

$$(f \smile g)[F_0 < \cdots < F_{p+q}] = f[F_0 < \cdots < F_p]\, g[F_p < \cdots < F_{p+q}]. \tag{8.2}$$

This is the Alexander–Whitney diagonal construction. It induces the topological cup product, because that chain diagonal is a chain approximation of the space diagonal; this is the usual simplicial cup-product theorem [3, Section 3.2]. Homotopy invariance and the proved retract identify its ring with the complement cohomology ring.

For local coefficients, let $\mathcal{L}$ be a local system of abelian groups on the complement, restricted to the model. Denote by L_F its fiber at the vertex associated to F, and by $\tau_{F,G} : L_F \to L_G$ transport along the edge $F < G$. Put

$$C_k(S;\mathcal{L}) = \bigoplus_{F_0 < \cdots < F_k} L_{F_0}.$$

For a coefficient a on a simplex of dimension $k \geq 1$, the boundary is

$$\partial(a[F_0 < \cdots < F_k]) = \tau_{F_0,F_1}(a)[F_1 < \cdots < F_k] + \sum_{j=1}^{k} (-1)^j a[F_0 < \cdots < \widehat{F}_j < \cdots < F_k].$$

The boundary in degree zero is zero. For a triangle, transport directly along $F_0 < F_2$ equals the composition along $F_0 < F_1 < F_2$, since the two paths are homotopic in that simplex. The double-deletion cancellation therefore proves $\partial^2 = 0$ also with local coefficients. For cohomology put $C^k(S;\mathcal{L}) = \prod_{F_0 < \cdots < F_k} L_{F_0}$. Cochains with values at the initial vertex have

$$(d\phi)[F_0 < \cdots < F_{k+1}] = \tau_{F_1,F_0}\phi[F_1 < \cdots < F_{k+1}] + \sum_{j=1}^{k+1} (-1)^j \phi[F_0 < \cdots < \widehat{F}_j < \cdots < F_{k+1}],$$

where τ_{F_1,F_0} is inverse edge transport. These are the simplicial local-coefficient complexes on this triangulation. To identify them with the complement complexes, let $i : S \hookrightarrow M(\mathcal{A})$ and $r : M(\mathcal{A}) \to S$ be the inclusion and retraction. The retraction homotopy supplies, by transport along its paths, an isomorphism $\mathcal{L} \cong r^* i^* \mathcal{L}$. Homotopy invariance with these identifications gives inverse maps on local homology and cohomology. Thus both displayed complexes compute the corresponding groups of $M(\mathcal{A})$ with coefficients in $\mathcal{L}$.

For each connected component, choose a spanning tree in the one-skeleton. Each non-tree oriented edge gives a generator of the fundamental group, and each two-simplex gives the relation of its oriented boundary after tree edges are contracted. The cellular van Kampen theorem proves that this is a presentation of the fundamental group; higher cells do not change it. Thus a specified finite-index subgroup can be used to lift every simplex and its incidence data, producing a finite model of the corresponding finite cover. More intrinsically, the strong deformation retraction lifts to any covering by the homotopy lifting property and retracts the cover onto the lifted finite complex. Because the original homotopy fixes S pointwise, its lift fixes the preimage of S pointwise as well. The lifted homotopy is therefore a strong deformation retraction, and its simplex incidences describe the cover of the complement.

9. Extensions and finite chain models

For the two-circle arrangement, the quotient L_0/DL_0 records torsion in the associated graded group of a torsion-free relative cohomology group. Passing directly to intersection cohomology loses this extension. Theorem 1.1 identifies the coefficient rings over which that loss disappears uniformly, and Proposition 4.1 identifies sufficient inverted primes for each fixed arrangement. Gorinov–Zakharov's integral binomial models retain the additional compatibility data; their connected-intersection theorem [2, Theorem 4] concerns a different hypothesis, since $C_0 \cap C_m$ has m components.

The finite model of Theorem 1.2 retains the integer boundaries themselves. Its defining data are the lattice maps, phases, and real face incidences. It therefore applies directly to the general elliptic equations in [1, Remark 2.3], with the universal-cover construction proposed in their Remark 3.12. For complex-multiplication arrangements, the endomorphism modules and component groups studied in [4] enter through the faithful blocks of Section 5. The finite chains, their diagonal, and their lifted incidences compute homology, cup products, and local coefficients from those equations.

References

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