INTEGRAL HOMOLOGY OF MILNOR FIBER BOUNDARIES

SUGAWARA'S CONJECTURE 1.4 AND THE TORSION PROBLEM OF NÉMETHI–SZILÁRD

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Abstract. For every complex projective line arrangement, we express the entire torsion subgroup of the first homology of its Milnor fiber boundary as the middle homology of a finite incidence complex. The two maps are a modular Orlik–Solomon gradient and local congruence defects. The formula preserves all prime-power invariant factors and gives a closed torsion-order formula with a modular resonance correction. It answers the torsion-formula problem of Némethi–Szilárd and proves all three assertions of Sugawara's Conjecture 1.4: the nonresonant formula, the covering-degree bound on torsion orders, and the pencil or near-pencil characterization of torsion-freeness. The last assertion follows from a peripheral-cover presentation, a modular lower bound, and complementary large-point and small-point incidence estimates.

1. The open problems and a finite incidence formula

Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be $n \geq 2$ distinct projective lines in $\mathbb{P}^2(\mathbb{C})$, and write

$$U = \mathbb{P}^2 \setminus \bigcup_i L_i.$$

Let F be the reduced Milnor fibre of the corresponding central arrangement in $\mathbb{C}^3$, and let ∂F denote its compact Milnor boundary. The set $\mathcal{X}$ comprises all intersection points, including double points. For $x \in \mathcal{X}$ put

$$I_x = \{i : x \in L_i\}, \qquad m_x = |I_x|, \qquad d_x = \gcd(m_x, n), \qquad R = \mathbb{Z}/n\mathbb{Z}.$$

Let Γ be the full bipartite incidence graph with line vertices, point vertices, and edges (i, x) for $i \in I_x$, oriented from lines to points. It is connected. Its cycle module over R is

$$Z_1(\Gamma; R) = \left\{ (c_{ix}) : \sum_{x \in L_i} c_{ix} = 0 \text{ for every } i, \quad \sum_{i \in I_x} c_{ix} = 0 \text{ for every } x \right\}. \tag{1.1}$$

It is free of rank

$$b = \sum_x (m_x - 1) - n + 1 = b_2(U; \mathbb{Z}). \tag{1.2}$$

The ordinary arrangement cohomology [3] gives $b_1(U) = n - 1$ and

$$\chi(U) = b - n + 2. \tag{1.3}$$

Define finite modules

$$L_n = \ker(\Sigma : R^n \to R)/R\mathbf{1}, \qquad C_x = \ker(\Sigma : (\mathbb{Z}/d_x)^{I_x} \to \mathbb{Z}/d_x)/(\mathbb{Z}/d_x)\mathbf{1}. \tag{1.4}$$

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The constant vectors lie in the respective sum-zero modules because $n = 0$ in R and $d_x \mid m_x$. Each is a primitive free cyclic summand, as projection to one coordinate takes $\mathbf{1}$ to 1. Consequently $L_n \cong R^{n-2}$ and $C_x \cong (\mathbb{Z}/d_x)^{m_x-2}$. At a double point, $C_x = 0$.

The two maps in our formula are

$$\Delta : L_n \longrightarrow Z_1(\Gamma; R), \qquad (\Delta a)_{ix} = m_x a_i - \sum_{j \in I_x} a_j, \tag{1.5}$$

$$\rho : Z_1(\Gamma; R) \longrightarrow \bigoplus_x C_x, \qquad \rho(c)_x = [(c_{ix} \bmod d_x)_{i \in I_x}]. \tag{1.6}$$

Every pair of lines meets at exactly one point, so

$$\sum_{x \in L_i} (\Delta a)_{ix} = n a_i - \sum_j a_j = 0, \qquad \sum_{i \in I_x} (\Delta a)_{ix} = 0.$$

Constant potentials map to zero. Differences of incident values of Δa are $m_x(a_i - a_j)$, hence zero modulo d_x. Thus these are well-defined maps and $\rho\Delta = 0$.

Theorem 1.1 (Complete torsion formula)**.** *For every complex projective line arrangement,*

$$\operatorname{Tor} H_1(\partial F; \mathbb{Z}) \cong \ker \rho / \operatorname{im} \Delta. \tag{1.7}$$

Equivalently, the entire finite group is the middle homology of

$$R^{n-2} \xrightarrow{\Delta} R^b \xrightarrow{\rho} \bigoplus_x (\mathbb{Z}/d_x)^{m_x-2},$$

with the basis-free maps (1.5)–(1.6). If $B_n = \ker \Delta$, its order is

$$|\operatorname{Tor} H_1(\partial F; \mathbb{Z})| = \frac{n^{\chi(U)} |B_n|^2}{\prod_x d_x^{m_x-2}}. \tag{1.8}$$

The free rank is

$$\operatorname{rank} H_1(\partial F; \mathbb{Z}) = \sum_x [1 + (m_x - 2) d_x]. \tag{1.9}$$

Némethi–Szilárd ask for a torsion formula in [2, Problem 24.4.19]; Suciu repeats the problem in [4, Problem 7.12], and Sugawara in [5, Problem 1.2]. Theorem 1.1 provides a uniform formula in terms of incidence modules and their maps. The known free-rank formula (1.9) is recovered in its proof, while (1.7) determines every torsion invariant factor.

Sugawara's Conjecture 1.4 in [5] consists of the following three assertions, proved in Sections 4–8.

Theorem 1.2. *For a central arrangement of n distinct planes in $\mathbb{C}^3$:*

1. *If $(m_x - 2)(\gcd(m_x, n) - 1) = 0$ for every intersection point, then*
$$\operatorname{Tor} H_1(\partial F; \mathbb{Z}) \cong (\mathbb{Z}/n)^{\chi(U)}.$$
2. *The deck group acts trivially on this torsion subgroup, and $n \operatorname{Tor} H_1(\partial F; \mathbb{Z}) = 0$. In particular, a cyclic subgroup of order m can occur only if $m \mid n$.*
3. *$H_1(\partial F; \mathbb{Z})$ is torsion-free if and only if the projective arrangement is a pencil or a near-pencil.*

Here a *pencil* has all lines through one point, and a *near-pencil* has $n - 1$ lines through one point and a remaining line avoiding it. Two projective lines form a pencil. Three form either a pencil or a triangle, which is a near-pencil. A single plane has boundary S^3 and is handled separately.

2. The boundary covering and its peripheral lattices

Blow up $\mathbb{P}^2$ at every $x \in \mathcal{X}$, including double points. The reduced total transform is a normal-crossing union of rational curves with dual graph Γ. Its strict-transform line has multiplicity 1, and the exceptional curve over x has multiplicity m_x. Let Y be the boundary of a regular neighbourhood of this divisor. The Milnor boundary is the cyclic n-cover $Y_n \to Y$ with meridian character 1 on lines and m_x on exceptional components; see [6, Propositions 2.2–2.3]. In the standard plumbing presentation its character is zero on the graph-cycle generators. This identification follows as well by restricting the homogeneous Milnor covering of the projective complement to its boundary collar.

The plumbing decomposes Y into circle bundles over punctured spheres. At each edge the gluing interchanges the fibre of one component with a peripheral curve of the other. Every vertex piece and edge torus has connected preimage: its fundamental group maps onto R, through the fibre at a line vertex and through any peripheral curve at a point vertex. The covering decomposition therefore has the same graph Γ.

Let P_0 be the sum of the homologies of the base vertex pieces, and P_1 the sum, in the covered pieces, of the subgroups generated by their boundary tori. Pushforward embeds P_1 as a finite-index lattice in P_0. Each base surface has nonempty boundary, so its circle bundle is trivial. Its first homology is the direct sum of the fibre class and the surface homology. Boundary circles span a primitive subgroup of the latter: quotienting gives the homology of the capped surface. Thus the quotient by the fibre and boundary classes is free. Pushforward is injective on this subgroup by the lattice calculations below. Let E_0 and E_1 be the corresponding sums of edge-torus homologies. Pushforward embeds E_1 in E_0 with index n at each edge, and

$$E_0/E_1 \cong R^{\mathrm{Edges}(\Gamma)}.$$

Write $j_0 : E_0 \to P_0$ and $j_1 : E_1 \to P_1$ for the signed inclusion maps. All pushforward squares commute.

2.1. **Line and point pieces.** On a line piece the fibre character already surjects onto R. The cover has fibre degree n and unchanged base, so no genus classes occur. Its local fundamental group is the product of a free group with the fibre group $\mathbb{Z}$. For any base loop, a fibre power cancels its character; the covering subgroup therefore abelianizes onto precisely the kernel of the character on homology. Its pushforward lattice is $\ker(P_{0,i} \to R)$, with quotient R.

For a point of multiplicity m, put $d = \gcd(m, n)$ and $h = n/d$. Quotienting the deck group by the fibre image gives a degree-d cover of the base surface. Each peripheral has character 1 in this quotient, so every boundary has a single degree-d preimage. There are exactly m boundary circles, and Euler characteristic gives genus

$$g_x = \frac{(m-2)(d-1)}{2}. \tag{2.1}$$

The base peripheral classes $b_1, \dots, b_m$ form a basis of its first homology: the fibre is $f = \sum_i b_i$, because the exceptional curve has self-intersection -1. Choose an integer a with $am \equiv d \pmod n$. The pushforwards of the covered fibre and peripherals generate the lattice

$$P_{1,x} = \langle h\mathbf{1},\ de_i - a\mathbf{1}\ (1 \le i \le m) \rangle \subseteq \mathbb{Z}^m. \tag{2.2}$$

The fibre degree is h; a peripheral needs d turns around the base and the indicated fibre adjustment to have trivial character.

Lemma 2.1. *The lattice* (2.2) *is exactly*

$$\left\{ c \in \mathbb{Z}^m : \sum_i c_i \equiv 0 \pmod n, \quad c_i \equiv c_j \pmod d \text{ for all } i, j \right\}. \tag{2.3}$$

Its index is nd^{m-2}.

Proof. Every generator satisfies the congruences. The congruence lattice contains $n\mathbb{Z}^m$, so its index can be computed modulo n. Write $c_i = r + dk_i$, with $r \bmod d$ and $k_i \bmod h$. The sum condition is $\sum_i k_i = -(m/d)r \bmod h$. There are dh^{m-1} admissible vectors modulo n, giving index $n^m/(dh^{m-1}) = nd^{m-2}$. The maximal minors of the generator matrix in (2.2) are $d^{m-1}(d-am)$ and, up to sign, hd^{m-1}. Since $n \mid d - am$, their greatest common divisor is $hd^{m-1} = nd^{m-2}$. The two lattices therefore have the same index and are equal. □

2.2. **The base has torsion-free homology.** Let W send each strict-transform line to $h - \sum_{x\in L_i} e_x$ and each exceptional curve to e_x, in H_2 of the blown-up plane. In these bases the intersection matrix is

$$Q = W^{\mathsf{T}} J W, \qquad J = \operatorname{diag}(1, -1, \dots, -1). \tag{2.4}$$

The map W is onto: exceptional columns give all e_x and any line column then gives h. Thus JW is onto and $\operatorname{im} Q = \operatorname{im} W^{\mathsf{T}}$. A right inverse of W over $\mathbb{Z}$ dualizes to a left inverse of W^{T}, so this image is primitive. Thus the plumbing cokernel is free, as is its additional graph-cycle summand. In particular, $H_1(Y;\mathbb{Z})$ is torsion-free and $\operatorname{coker} j_0$ is free of rank $n-1$.

Peripheral and edge pushforwards are rational isomorphisms. Consequently $\operatorname{coker} j_1 \to \operatorname{coker} j_0$ is a rational isomorphism with free target. Every torsion class lies in its kernel, and every kernel class is zero after tensoring with $\mathbb{Q}$, hence is torsion. The covered-piece quotients by P_1 and the graph-cycle quotient in Mayer–Vietoris are free. Hence

$$\operatorname{Tor} H_1(Y_n;\mathbb{Z}) = \ker(\operatorname{coker} j_1 \to \operatorname{coker} j_0). \tag{2.5}$$

The free rank of $\operatorname{coker} j_1$ is $n-1$. Adding the covered genera and graph cycles gives

$$\operatorname{rank} H_1(Y_n;\mathbb{Z}) = n - 1 + b + \sum_x (m_x - 2)(d_x - 1).$$

Using (1.2) simplifies this expression to (1.9).

3. Passing to the finite incidence complex

The inclusion square of lattices induces

$$\psi : E_0/E_1 \longrightarrow P_0/P_1.$$

An edge value $c_{ix} \in R$ can be represented by c_{ix} times the adjacent line fibre. At its line endpoint its character is c_{ix}, and at its point endpoint its pushforward is $c_{ix}b_i$. The line condition and Lemma 2.1 therefore say

$$\psi(c) = 0 \quad\Longleftrightarrow\quad c \in Z_1(\Gamma;R), \quad \rho(c) = 0.$$

Thus $\ker\psi = \ker\rho$.

Lemma 3.1. *There is an isomorphism*

$$\ker(\operatorname{coker} j_1 \to \operatorname{coker} j_0) \cong \ker\psi/\operatorname{im}(\ker j_0 \to E_0/E_1).$$

Proof. If $e \in E_0$ represents a class in $\ker\psi$, then $j_0(e) \in P_1$ and defines a class in $\operatorname{coker} j_1$ whose image in $\operatorname{coker} j_0$ is zero. Every kernel class arises this way. It is zero precisely when $j_0(e) = j_0(e_1)$ for some $e_1 \in E_1$, or $e \in E_1 + \ker j_0$. Quotienting proves the assertion. □

We now identify the remaining image. On edge (i,x) choose coordinates $t_{ix}f_i + s_{ix}f_x$ in the two component fibres, and use a positive inclusion at the line and a negative inclusion at the point. Put $e_i = 1 - \#\{x : x \in L_i\}$ for the strict-transform self-intersection. Each vertex has the single relation $e_v f_v + \sum b_{ve} = 0$, with $e_x = -1$ at a point. A kernel vector of j_0 must therefore satisfy

$$s_{ix} = \lambda_i, \qquad \sum_x t_{ix} = e_i\lambda_i, \qquad -t_{ix} = \lambda_x, \qquad -\sum_i s_{ix} = -\lambda_x.$$

The resulting consistency conditions are $Q\lambda = 0$. By (2.4) and injectivity of W^{T}, these are exactly $W\lambda = 0$, or

$$\sum_i \lambda_i = 0, \qquad \lambda_x = \sum_{i \in I_x} \lambda_i. \tag{3.1}$$

The covering character on that vector is $t_{ix} + m_x s_{ix} = m_x\lambda_i - \lambda_x$. Therefore its image modulo n is exactly $\operatorname{im}\Delta$. Every sum-zero vector modulo n admits an integral sum-zero lift, by adjusting one entry by a multiple of n. Constants give zero image, accounting for the quotient in L_n. Lemma 3.1 and (2.5) prove (1.7).

Remark 3.2. For an intermediate N-fold boundary cover with $N \mid n$, the same covering-lattice calculation applies with $R = \mathbb{Z}/N$, $d_x = \gcd(m_x, N)$, and $L_N = \ker(\Sigma : R^n \to R)/R\mathbf{1}$. The module L_N is free of rank $n-2$ because $N \mid n$. Each edge pushforward has index N, and the point lattice has index $Nd_x^{m_x-2}$, so the proof gives $\operatorname{Tor} H_1(Y_N;\mathbb{Z}) \cong \ker\rho_N/\operatorname{im}\Delta_N$ directly.

4. Duality, resonance, and the torsion order

The module $B_n = \ker\Delta$ has the explicit description

$$B_n = \left\{ a \in R^n : \sum_i a_i = 0, \quad m_x a_i = \sum_{j \in I_x} a_j \text{ for all } i \in I_x \right\} / R\mathbf{1}. \tag{4.1}$$

For every $N \mid n$, define B_N by the same formula (4.1) with coefficient ring $\mathbb{Z}/N$; equivalently it is the kernel of the coefficient version of Δ in Remark 3.2. In particular B_p and B_{p^a} below refer to these coefficient kernels. The module B_n is the first Aomoto cohomology of the projective Orlik–Solomon algebra over R in the all-ones direction. To check this description directly, use the integral degree-two pencil basis at each point: the coefficients of $\mathbf{1} \wedge a$ are the expressions in (1.5). The degree-one projective module is the sum-zero module, and the degree-zero image is $R\mathbf{1}$, since $n = 0$ in R. These integral coefficients also describe points whose multiplicity is a nonunit in R.

Lemma 4.1. *Pontryagin duality gives*

$$\operatorname{Hom}(\operatorname{coker}\rho, \mathbb{Q}/\mathbb{Z}) \cong B_n.$$

Proof. Each C_x has a perfect pairing with itself, the dot product of representatives divided by d_x, valued in $\mathbb{Q}/\mathbb{Z}$. Constants are annihilated by sum-zero vectors, and the orthogonal complement of the sum-zero module is the constant module; thus the induced pairing is perfect, also for composite d_x.

A tuple $r_x \in C_x$ pulls back along ρ to the edge cochain $(n/d_x)r_{ix}$ over R. It annihilates all graph cycles precisely when it is a vertex gradient. This characterization holds over R by a spanning-tree basis of cycles. Write its potentials as u_i on lines and u_x on points, so

$$u_x - u_i = (n/d_x)r_{ix} \pmod{n}. \tag{4.2}$$

Such sum-zero r_x exist exactly when the incident u_i agree modulo n/d_x and $m_x u_x = \sum_{i \in I_x} u_i$ modulo n. These conditions are equivalent to $m_x u_i = \sum_{j \in I_x} u_j$ for all incident i: the annihilator of m_x in R consists of multiples of n/d_x. For the converse, choose $u_x = u_{i_0}$ at one incident line and put $r_{ix} = (u_x - u_i)/(n/d_x)$ in $\mathbb{Z}/d_x$. The incident differences are divisible by n/d_x, and their sum is zero modulo n, so $\sum_i r_{ix} = 0$ modulo d_x. The line-divergence identity then gives $0 = \sum_{x \in L_i}(m_x u_i - \sum_{j \in I_x} u_j) = nu_i - \sum_j u_j$, proving $\sum_j u_j = 0$ in R. Changing u_x by n/d_x changes r_x by a constant, already factored out in C_x. A common change of every vertex potential changes no gradient. If every r_x class vanishes, all incident line potentials agree, and connectedness makes them globally constant. Conversely any Δ-cocycle gives these gradients. This establishes the stated bijection with (4.1). □

Taking orders in the finite complex gives

$$|\ker \rho| = \frac{|Z_1|\,|\operatorname{coker}\rho|}{|\bigoplus_x C_x|}, \qquad |\operatorname{im}\Delta| = \frac{|L_n|}{|B_n|}.$$

Lemma 4.1, $|Z_1| = n^b$, and $|L_n| = n^{n-2}$ now give (1.8). This completes the proof of Theorem 1.1.

4.1. **The nonresonant formula.** If $(m_x-2)(d_x-1) = 0$ for all x, then every C_x is zero. Lemma 4.1 gives $B_n = 0$. Thus Δ is an injection of free R-modules. Over each local factor $\mathbb{Z}/p^a$ of R, Smith form has only unit pivots: a nonunit pivot would have a nonzero annihilator and yield a kernel. The injection therefore splits, and its cokernel is free of rank $b-(n-2) = \chi(U)$. This proves Theorem 1.2(1).

More generally, write $a = v_p(n)$ and $b'_x = \min(a, v_p(m_x))$ for $p \mid n$.

Corollary 4.2. *If $B_p = 0$, then*

$$\operatorname{Tor} H_1(\partial F;\mathbb{Z})_p \cong (\mathbb{Z}/p^a)^{\chi(U)-r_p} \oplus \bigoplus_{x:p\mid m_x} (\mathbb{Z}/p^{a-b'_x})^{m_x-2}, \quad r_p = \sum_{x:p\mid m_x}(m_x-2).$$

Terms of exponent zero are omitted.

Proof. Smith form of an integer matrix for Δ shows that $B_p = 0$ implies $B_{p^a} = 0$. Hence Δ splits over $\mathbb{Z}/p^a$, leaving a free middle quotient of rank $\chi(U)$. Lemma 4.1 says that the induced ρ is surjective. Its target is $\bigoplus_x(\mathbb{Z}/p^{b'_x})^{m_x-2}$. Smith form of this surjection makes its kernel the displayed group; the unused coordinates give the $\chi(U)-r_p$ free factors. In particular this number is nonnegative. □

5. Deck invariance and the exponent bound

The deck action on the peripheral subgroups gives both its triviality on torsion and the transfer proof of the exponent bound.

For each covered vertex piece, let P_v be the subgroup generated by its boundary tori. Its quotient is the free homology of the capped base surface. Every deck transformation preserves each covered boundary torus, since its preimage is connected, and acts on it by a translation. It therefore fixes its integral homology and consequently every element of P_v. The gluing image lies in $P = \bigoplus_v P_v$. The two exact sequences

$$0 \to P/\operatorname{im} j_1 \to (\textstyle\bigoplus_v H_1(V_v))/\operatorname{im} j_1 \to (\bigoplus_v H_1(V_v))/P \to 0$$

and

$$0 \to (\textstyle\bigoplus_v H_1(V_v))/\operatorname{im} j_1 \to H_1(Y_n;\mathbb{Z}) \to H_1(\Gamma;\mathbb{Z}) \to 0$$

have free final terms. Thus the entire torsion lies in the pointwise fixed quotient $P/\operatorname{im} j_1$. For a torsion element z, pushforward to $H_1(Y;\mathbb{Z})$ is zero because the latter is free. Integral transfer satisfies

$$\operatorname{tr}\, q_*z = \sum_{g\in C_n} g_*z = nz.$$

It follows that $nz = 0$, proving Theorem 1.2(2).

6. A peripheral presentation

To prove the torsion-free characterization, we use a presentation in which every covering coefficient remains integral. Blow up only points of multiplicity at least three, leaving double points as ordinary normal crossings. Let t be the number of blown points, E the number of edges of this graph, and $V = n+t$ its number of vertices. Set

$$c = \chi(U) = E - 2n - t + 3. \tag{6.1}$$

At a multiple point put $d_x = \gcd(m_x, n)$, $h_x = n/d_x$, and $g_x = m_x/d_x$. The notation g_x in this section is a covering coefficient; the genus in (2.1) will be written γ_x.

Proposition 6.1. *The potentially torsion-bearing part P of $H_1(\partial F;\mathbb{Z})$ has generators F_i, G_x, q_{xi}, and q_{ij}, with relations*

$$h_x q_{xi} + g_x F_i - G_x = 0 \quad (i \in I_x), \tag{6.2}$$

$$\sum_{i\in I_x} q_{xi} = 0, \qquad \sum_{i\in I_x} F_i - d_x G_x = 0, \tag{6.3}$$

$$F_j - F_i = n q_{ij} \quad \text{at a double point}, \tag{6.4}$$

$$F_i + \sum_{e \text{ outward from } i} q_e = 0 \quad \text{on each line}. \tag{6.5}$$

Double edges have opposite signs at their ends; a point-edge q_{xi} is outward from its line. Furthermore,

$$\operatorname{rank} P = n - 1, \qquad H_1(\partial F;\mathbb{Z}) \cong P \oplus \mathbb{Z}^{b_1(\Gamma)+2\sum_x \gamma_x}.$$

The splitting as abelian groups need not be deck-equivariant.

Proof. In the uncovered edge basis put $F_i = n f_i$, $G_x = (n/d_x) f_x$, and $q_{xi} = f_x - m_x f_i$. The two generators of its covering lattice give (6.2). Choose a with $a m_x \equiv d_x \pmod n$ and set $c_x = (d_x - a m_x)/n$. The lifted point peripheral is

$$p_i = d_x f_i - a f_x = -a q_{xi} + c_x F_i.$$

Its relation is $\sum_i p_i - c_x d_x G_x = 0$. Put $S = \sum_i q_{xi}$ and $A = \sum_i F_i - d_x G_x$. The peripheral relation and the sum of (6.2) become $-aS + c_x A = 0$ and $h_x S + g_x A = 0$. Their coefficient matrix has determinant

$$(n/d_x) c_x + (m_x/d_x) a = 1.$$

It is unimodular, giving (6.3) integrally. At a double edge the fibre difference gives (6.4). At a line, write r_i for the number of blown points on it. Its self-intersection is $1 - r_i$. Pair counting on that line gives

$$\sum_{x\in L_i,\, m_x \ge 3} (m_x - 1) + \#\{\text{double points on } L_i\} = n - 1.$$

Thus the self-intersection plus adjacent divisor multiplicities is n. Substituting $f_x = q_{xi} + m_x f_i$ at point edges and $f_j = q_{ij} + f_i$ at double edges into the line relation gives $n f_i + \sum_e q_e = 0$, which is (6.5).

The quotients by the peripheral subgroups and the graph-cycle quotient are free, yielding the asserted abelian splitting. Their pushforwards are rational isomorphisms, as in Section 2; the gluing cokernel has the rational rank of the base plumbing cokernel. The matrix $W^{\mathsf T} J W$ has rank $t+1$, so this rank is $V - (t+1) = n - 1$. □

7. A modular lower bound at every multiple point

For $p \mid n$ write $a = v_p(n)$, and let

$$\tau_p = \dim_{\mathbb{F}_p}(\operatorname{Tor} H_1(\partial F;\mathbb{Z}) \otimes \mathbb{F}_p).$$

A blown point is called *high for p* if $v_p(m_x) \ge a$. This designation excludes unblown double points.

Proposition 7.1. *For every prime $p \mid n$,*

$$\tau_p \ge c - \sum_{x \text{ high for } p} (m_x - 2). \tag{7.1}$$

Proof. By Proposition 6.1, $\dim_{\mathbb{F}_p}(P \otimes \mathbb{F}_p) = n - 1 + \tau_p$. Its dual is the scalar solution space of (6.2)–(6.5) over $k = \mathbb{F}_p$. Consider the following auxiliary incidence space: at each blown point require all incident F_i to equal a new vertex value F_x, and require $\sum_i q_{xi} = 0$. Retain the double-edge equalities $F_j = F_i$ and all line equations.

The first requirements make every vertex value equal a common f, since the graph is connected. The remaining equations say that the edge flow has divergence $-f$ on each line and zero on point vertices, up to the fixed incidence orientation. A flow with these divergences exists because their total is $-nf = 0$ in k. The incidence map of a connected graph surjects onto sum-zero vertex vectors. Each such solution is a translate of its cycle space of dimension $E - V + 1$. Thus the auxiliary solution space has dimension

$$E - V + 2 = (n - 1) + c. \tag{7.2}$$

At a non-high point, $h_x = 0$ in k and g_x is a unit. The edge equations require $F_i = G_x/g_x$ for every incident i. The second sum relation is automatic since $m_x/g_x = d_x$, and the first is exactly $\sum_i q_{xi} = 0$. Thus the auxiliary relations coincide with the presentation's relations there after renaming the vertex value.

At a high point, h_x is a unit and $d_x = m_x = 0$ in k. Recovering G_x from one edge reduces the relations to

$$\sum_i F_i = 0, \qquad q_{xi} + (g_x/h_x)F_i \text{ is independent of } i.$$

The relation $\sum_i q_{xi} = 0$ also follows from these equations: sum the edge equations and use $m_x = 0$ and $\sum_i F_i = 0$. In the auxiliary space all F_i equal f, so the first condition holds. It suffices to require all incident q_{xi} equal. The auxiliary incident vector lies in the $(m_x - 1)$-dimensional sum-zero space. Its constant subspace has dimension one and lies in that space because $m_x = 0$ in k. These additional restrictions lose at most $m_x - 2$ dimensions. Impose them at every high point. The resulting space maps into the presentation's solution space by retaining all F_i, q_e and recovering each G_x uniquely. This map is injective: every auxiliary point value was already fixed by its incident line values, so the retained coordinates determine the auxiliary solution. Combining its dimension bound with (7.2) proves (7.1). □

8. Torsion-freeness: the two incidence regimes

Assume the arrangement is neither a pencil nor a near-pencil. Then $n \geq 4$ and every multiplicity is at most $n - 2$. Let M be the largest multiplicity and set $k = n - M \geq 2$. We first record two elementary estimates.

Lemma 8.1. *If $M \geq 3$, then*

$$c \geq (M - 2)(k - 1). \tag{8.1}$$

If x has multiplicity M, any collection $\mathcal{B}$ of multiple points excluding x satisfies

$$\sum_{y \in \mathcal{B}} (m_y - 2) \leq \binom{k}{2}. \tag{8.2}$$

If all points are double, $c = (n - 2)(n - 3)/2 > 0$.

Proof. Choose a line through x as infinity. The other $M-1$ lines through x become parallel affine lines. Each of the remaining k lines meets all of them. A point where r remaining lines meet one parallel line contributes r to affine b_2. All $(M-1)k$ such crossings are thus counted, and other contributions are nonnegative. Since affine $b_1 = n - 1$, this gives $c \geq 1 - (n - 1) + (M - 1)k = (M - 2)(k - 1)$.

A point $y \neq x$ lies on at most one line through x. If r of the remaining k lines pass through it, then $m_y = r$ or $r + 1$. For a multiple point $r \geq 2$ and $m_y - 2 \leq \binom{r}{2}$. The pairs of remaining lines counted at distinct points are disjoint, proving (8.2). The double-point formula follows from (1.2)–(1.3). □

8.1. **A large point.** Suppose $M > (n+4)/3$. Since $0 < M < n$, some prime $p \mid n$ has $v_p(M) < v_p(n)$; otherwise $n \mid M$. The chosen largest point is non-high for this prime. By Proposition 7.1 and Lemma 8.1,

$$\tau_p \geq (M-2)(k-1) - \binom{k}{2} > 0,$$

because $M - 2 > k/2$ is precisely $3M > n + 4$. This proves torsion exists in the large-point regime.

8.2. **Small points and the complex incidence inequality.** Now suppose $M \leq (n+4)/3$. Let t_m be the number of points of multiplicity m, let $s = \sum_m t_m$, and put $I = \sum_m m t_m$. For $n \geq 4$ this regime gives $M \leq 2n/3$. Langer's complex line-arrangement inequality [1, Proposition 11.3.1] applies and gives

$$I \geq n(n+3)/3, \qquad \sum_m m(m-1)t_m = n(n-1). \tag{8.3}$$

Cauchy–Schwarz and monotonicity of $u^2/(n(n-1)+u)$ for $u > 0$ give

$$s \geq \frac{I^2}{n(n-1)+I} \geq \frac{(n+3)^2}{12}. \tag{8.4}$$

For $n \geq 15$ the right side is at least $2n - 3$. Put $R_0 = \sum_{m \geq 3}(m-2)t_m$. Then

$$c = R_0 + s - 2n + 3 \geq R_0. \tag{8.5}$$

If $R_0 = 0$, Proposition 7.1 gives $\tau_p \geq c > 0$ for any $p \mid n$. Otherwise choose a multiple point of multiplicity $m < n$ and a prime $p \mid n$ with $v_p(m) < v_p(n)$. Its loss is absent from the high-point sum, so

$$\tau_p \geq c - \sum_{x \text{ high}} (m_x - 2) \geq R_0 - [R_0 - (m-2)] > 0.$$

This covers every $n \geq 15$ in the small-point regime.

8.3. **The remaining degrees.** If $n = p^a$, no proper multiplicity $m < n$ is high for p. Lemma 8.1 gives $c > 0$, so $\tau_p \geq c > 0$. Below 15 the remaining non-prime-power degrees are $6, 10, 12, 14$. Under the small-point bound, at $n = 6$ choose $p = 2$ (then $M \leq 3$), at $n = 10$ choose $p = 5$ (then $M \leq 4$), and at $n = 14$ choose $p = 7$ (then $M \leq 6$). No multiple point is high in these cases.

At $n = 12$ one has $M \leq 5$. Put $x = t_3$, $y = t_4$, $z = t_5$. Pair counting and Euler characteristic give

$$t_2 + 3x + 6y + 10z = 66, \qquad c = 45 - x - 3y - 6z.$$

The lower bounds are $\tau_2 \geq c - 2y$ and $\tau_3 \geq c - x$. If neither were positive, then

$$2x + 3y + 6z \geq 45, \qquad x + 5y + 6z \geq 45,$$

so $3x + 8y + 12z \geq 90$. But (8.3) gives $I = 132 - 3x - 8y - 15z \geq 60$, hence $3x + 8y + 15z \leq 72$, a contradiction. Thus torsion exists also in this critical degree.

8.4. **Pencils and near-pencils.** For a pencil the incidence graph is a star, $Z_1 = 0$, and (1.7) gives zero torsion. For a near-pencil every local defect C_x is zero: its only higher multiplicity is $n - 1$, coprime to n. The nonresonant formula gives zero torsion because $\chi(U) = 0$. Together with the preceding two regimes, this proves Theorem 1.2(3) and completes its proof.

9. Invariant factors and examples

The maps (1.5)–(1.6) are finite incidence maps, so (1.7) can be evaluated directly by integer Smith reduction after presenting its finite modules. The order can also be read from an integer matrix: represent L_n by $a_n = 0$ and $\sum_i a_i = 0$, with free coordinates $a_1, \dots, a_{n-2}$ and $a_{n-1} = -\sum_{i \le n-2} a_i$. Form the integer matrix K whose rows are $m_x a_i - \sum_{j \in I_x} a_j$. It is injective over $\mathbb{Q}$: equalities on a point make all incident potentials equal, and connectedness and $a_n = 0$ make them zero. If its nonzero Smith factors are $s_1, \dots, s_{n-2}$, then

$$B_n \cong \bigoplus_{j=1}^{n-2} \mathbb{Z}/\gcd(n, s_j), \quad |\operatorname{Tor} H_1(\partial F; \mathbb{Z})| = \frac{n^{\chi(U)} \prod_j \gcd(n, s_j)^2}{\prod_x \gcd(n, m_x)^{m_x - 2}}.$$

The integral rows of K need not be graph cycles; their reduction modulo n is exactly Δ, which is all this computation uses.

For a generic arrangement every defect is zero. The answer is $(\mathbb{Z}/n)^{(n-2)(n-3)/2}$, recovering Sugawara's theorem [5, Theorem 1.3]. For a pencil, $B_n = L_n$ and the order formula reduces to one, even though $\chi(U) = 2 - n$ can be negative. The resonance factor cancels the negative power of n.

For the six-line A_3 arrangement there are four triple and three double points, with $\chi(U) = 2$. The all-ones cocycles over $\mathbb{F}_2$ are constant; over $\mathbb{F}_3$, equality on opposite double-point pairs leaves one independent class after the sum and constant relations. Thus $B_6 \cong \mathbb{Z}/3$, and (1.8) gives $6^2 \cdot 3^2/3^4 = 4$. The exponent bound and Corollary 4.2 at $p = 2$ give $\operatorname{Tor} H_1(\partial F; \mathbb{Z}) \cong (\mathbb{Z}/2)^2$. Here ρ is not surjective: its cokernel is $\mathbb{Z}/3$. This explains the resonance correction in the order formula.

For two four-line pencils with distinct centers and sixteen double cross-intersections, $n = 8$, $\chi(U) = 9$, and $B_2 = 0$: the complete bipartite double-point graph forces all mod-2 potentials to agree. At the two fourfold points, $a = 3$, $b'_x = 2$, and $r_2 = 4$. Corollary 4.2 gives

$$\operatorname{Tor} H_1(\partial F; \mathbb{Z}) \cong (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/8)^5,$$

whose order is 2^{19}. The finite complex records both exponents in this example. For resonant coefficient systems with $B_p \neq 0$, (1.7) computes the primary group by the two incidence maps; Corollary 4.2 treats the case $B_p = 0$.

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