ARITHMETIC OF PRIMITIVE TORIC COVERS

THE LAYER-POSET QUESTION IN DELUCCHI–MARMO'S REMARK 3.2.7, WITH EXACT COVER COUNTS AND POSITIVE DENSITY

DENZEL ZHENG

Abstract. We determine which primes admit a primitive connected ambient cover of a toric arrangement, and count these covers in every finite abelian deck-group type. At a fixed ambient dimension, the answers are determined by the layer poset. For essential arrangements the dimension is itself determined by that poset, giving an affirmative answer to the question in Delucchi–Marmo's Remark 3.2.7 in the essential range. A single-divisor example shows why the ambient dimension is essential to this determination. The prime-degree criterion needs only rank-two layer counts: their divisibility detects equality of reduced projective character lines. Higher independent intersections recover the entire mod-p rank function. We then derive elementary, cyclic, and arbitrary abelian cover counts, followed by a local generating function whose numerator is a matroid characteristic polynomial. Its global Euler product yields an asymptotic formula and a strictly positive density among all ambient torus covers in dimensions at least two. All covers are counted up to isomorphism over the given torus, without a deck marking; the full inverse image of every original divisor is required to remain connected.

1. The layer-poset question and the prime-degree answer

Let $\Lambda \cong \mathbb{Z}^d$ be a free character lattice and $T = \mathrm{Hom}(\Lambda, \mathbb{C}^*)$ its complex torus. Consider an arrangement

$$\mathcal{B} = \{H_i : 1 \le i \le n\}, \qquad H_i = \{t \in T : \chi_i(t) = b_i\}, \tag{1.1}$$

where $\chi_i \in \Lambda$ is nonzero and primitive, $b_i \in \mathbb{C}^*$, and the divisors H_i are distinct. Primitivity means that χ_i is not an integer multiple $m\psi$ with $|m| > 1$; equivalently, H_i is connected. We allow arbitrary translations, parallel divisors, and nonessential arrangements. The arrangement is essential if $\mathrm{rank}_{\mathbb{Q}}\langle \chi_1, \dots, \chi_n \rangle = d$.

The layer poset $L(\mathcal{B})$ consists of T and all connected components of nonempty intersections of divisors, ordered by reverse inclusion. Its rank is complex codimension. Its atoms are the divisors themselves.

Definition 1.1. A connected finite ambient torus cover $f : U \to T$ is *primitive for* $\mathcal{B}$ if every full inverse image $f^{-1}(H_i)$ is connected. A primitive p-cover is such a cover of degree p^a for some $a \ge 1$. Covers are identified by covering isomorphisms over T. A deck-group type means its abstract isomorphism type, with no specified marking.

The definition agrees with [1, Definitions 3.2.1, 3.2.4 and Remark 3.2.5]. Delucchi–Marmo's Theorem 3.2.6 gives a character criterion for primitive p-covers. Their Remark 3.2.7 asks whether the set of such primes is constant among arrangements with the same layer poset. Theorem 1.2 expresses the criterion in terms of rank-two intersections for arrangements in a fixed ambient dimension. The special case with total intersection $\{1\}$ is already identified in that remark through Pagaria's character-recovery results [3]; it also points to the arithmetic representation theory of [4]

Date: Research note / October 2026.

2020 *Mathematics Subject Classification.* Primary 52C35; Secondary 05B35, 11M41, 20K01, 57M10.

Key words and phrases. toric arrangement, primitive cover, layer poset, integer lattice, finite abelian deck group, cover-counting series, natural density.

for the general realization problem. The intersection counts below supply the reduced character data directly.

For distinct atoms, put

$$c_{ij} = \#\{W \in L(\mathcal{B}) : \operatorname{rank} W = 2,\ H_i \le W,\ H_j \le W\}. \tag{1.2}$$

For a prime p, define

$$i \sim_p j \quad \Longleftrightarrow \quad i = j \text{ or } p \mid c_{ij}. \tag{1.3}$$

As proved below, this is an equivalence relation. Let $k_p(\mathcal{B})$ be its number of classes, with $k_p = 0$ if $n = 0$. Set

$$L_d(p) = \frac{p^d - 1}{p - 1} = 1 + p + \cdots + p^{d-1} \quad (d \ge 1).$$

Theorem 1.2. *Let $d \ge 1$. A primitive p-cover exists if and only if*

$$k_p(\mathcal{B}) < L_d(p). \tag{1.4}$$

The number of primitive connected covers of degree exactly p is

$$L_d(p) - k_p(\mathcal{B}). \tag{1.5}$$

Thus the admissible prime set and its prime-degree counts are determined by d and the rank-at-most-two truncation of $L(\mathcal{B})$. For essential arrangements they are determined by $L(\mathcal{B})$ alone.

The stronger counts below also depend only on d and $L(\mathcal{B})$. Let $a_m(\mathcal{B})$ denote the number of primitive connected ambient covers of degree m, including $a_1 = 1$. For each prime p, write

$$r_p(S) = \operatorname{rank}_{\mathbb{F}_p}\{\overline{\chi}_i : i \in S\}, \qquad S \subseteq [n],$$

where $\overline{\chi}_i$ is the reduction of χ_i modulo p. Section 3 reconstructs this function from the layer poset. Put

$$P_{\mathcal{B},p}(z) = \sum_{S \subseteq [n]} (-1)^{|S|} z^{r_p(S)}. \tag{1.6}$$

Theorem 1.3. *For every prime p and $d \ge 1$, there is a formal power-series identity*

$$\sum_{k \ge 0} a_{p^k}(\mathcal{B}) z^k = \frac{P_{\mathcal{B},p}(z)}{\prod_{j=0}^{d-1}(1 - p^j z)}. \tag{1.7}$$

The sequence a_m is multiplicative. For $\operatorname{Re} s > d$,

$$\sum_{m \ge 1} \frac{a_m(\mathcal{B})}{m^s} = \left(\prod_{j=0}^{d-1} \zeta(s - j)\right) \prod_p P_{\mathcal{B},p}(p^{-s}). \tag{1.8}$$

If $d \ge 2$, put

$$D_{\mathcal{B}} = \prod_p P_{\mathcal{B},p}(p^{-d}). \tag{1.9}$$

Then $0 < D_{\mathcal{B}} \le 1$, and

$$\sum_{m \le X} a_m(\mathcal{B}) \sim \frac{\prod_{j=2}^{d} \zeta(j)}{d} D_{\mathcal{B}} X^d. \tag{1.10}$$

The ratio of this count to the count of all connected ambient torus covers of degree at most X tends to $D_{\mathcal{B}}$.

The density counts covering isomorphism classes individually, including distinct classes of the same degree. Throughout the paper, a cover means a connected finite cover of the ambient torus T.

2. Integer lattices and rank-two intersection counts

Lemma 2.1. *Connected finite ambient covers of T are in bijection with the finite-index overlattices*

$$\Lambda \subseteq \Lambda' \subseteq \Lambda \otimes \mathbb{Q}.$$

The cover is the restriction map $\mathrm{Hom}(\Lambda', \mathbb{C}^*) \to \mathrm{Hom}(\Lambda, \mathbb{C}^*)$*, its degree is* $[\Lambda' : \Lambda]$*, and its deck group is* $\mathrm{Hom}(\Lambda'/\Lambda, \mathbb{C}^*)$*. It is primitive for $\mathcal{B}$ exactly when every χ_i is primitive in Λ'.*

Proof. The cocharacter lattice is $N = \mathrm{Hom}(\Lambda, \mathbb{Z}) = \pi_1(T)$. Connected finite covers of T are classified by finite-index subgroups $N' \subseteq N$, since T retracts onto a real torus and N is abelian; this is the usual covering classification [2, Section 1.3]. The dual lattice $\Lambda' = \mathrm{Hom}(N', \mathbb{Z})$ embeds in $\Lambda \otimes \mathbb{Q}$ by extending each homomorphism rationally. Conversely an overlattice has as its dual such a subgroup N'. These operations are inverse, and dual integer bases show that their indices agree. The displayed restriction map has that induced subgroup on fundamental groups, so gives the corresponding cover. Its kernel is the stated character group.

In a basis of Λ', a character χ has coordinate gcd q. An integer change of basis writes it as q times a basis character. For every $b \in \mathbb{C}^*$, its equation is then $u_1^q = b$, with precisely q disjoint connected translates of a codimension-one subtorus. It is connected if and only if $q = 1$. Applying this to each inverse divisor proves the last assertion. □

We will repeatedly use the elementary determinantal consequence of Smith normal form. If M is an integer matrix of rank k, then the index of its row lattice in its saturation is the gcd of its k-rowed minors. Indeed, unimodular row and column transformations preserve the gcd of these minors by the Cauchy–Binet formula and its inverse. In Smith form both the index and the gcd are the product of the nonzero diagonal factors. A matrix of rank k remains of full rank modulo p exactly when this gcd is not divisible by p.

Lemma 2.2. *If χ_i, χ_j are rationally independent, $H_i \cap H_j$ is nonempty and c_{ij} is the index of $\mathbb{Z}\chi_i + \mathbb{Z}\chi_j$ in its saturation. If they are rationally dependent, $c_{ij} = 0$. For every prime p,*

$$[\overline{\chi}_i] = [\overline{\chi}_j] \quad \Longleftrightarrow \quad p \mid c_{ij}. \tag{2.1}$$

All reduced characters here are nonzero, and $p \mid 0$ has its usual meaning.

Proof. For the independent pair, the map $T \to (\mathbb{C}^*)^2$ defined by its characters is surjective. One may see this in Smith bases: it is $(u_1, \dots, u_d) \mapsto (u_1^{s_1}, u_2^{s_2})$, followed by automorphisms of source and target; power maps on $\mathbb{C}^*$ are surjective. Every translated fiber is nonempty with $s_1 s_2$ connected components, equal to the saturation index. Those components have codimension two and are exactly the layers counted in (1.2). The first Smith factor is one, since the coordinates of the primitive row χ_i already have gcd one. Hence $s_2 = c_{ij}$, and reduction has rank one exactly when $p \mid c_{ij}$. The two nonzero reduced rows then span the same line.

If the pair is rationally dependent, primitive lattice vectors on one rational line agree up to sign. The distinct divisors in that direction are disjoint translates, so $c_{ij} = 0$; their reductions span the same line for every prime. Finally, a primitive vector cannot have all its coordinates divisible by p, proving nonzero reduction. □

By (2.1), the classes of (1.3) are the distinct projective lines spanned by the reduced characters. In particular, $\sim_p$ is an equivalence relation and $k_p(\mathcal{B})$ is the number of those lines.

Proof of Theorem 1.2. An index-p overlattice lies in $p^{-1}\Lambda$ and is of the form

$$\Lambda' = \Lambda + \mathbb{Z}(v/p), \qquad 0 \neq \overline{v} \in \Lambda/p\Lambda.$$

It depends only on the line $\mathbb{F}_p\overline{v}$, and each line gives one overlattice. There are $L_d(p)$ such lines.

In a p-primary overlattice, an originally primitive character χ loses primitivity if and only if $\chi/p \in \Lambda'$. To prove this, if $\chi = q\psi$ for a prime q and $\psi \in \Lambda'$, then $\psi \notin \Lambda$ by original primitivity. Its

class has order q in Λ'/Λ, forcing $q = p$. Conversely divisibility by p is immediate from $\chi/p \in \Lambda'$. A composite divisor contains a prime divisor, so this also proves the claim for arbitrary nonprimitivity.

For index p, the forbidden condition is thus precisely $[\overline{\chi}_i] = [\overline{v}]$. Lemma 2.2 counts its distinct forbidden lines as $k_p(\mathcal{B})$. This proves (1.5) and the degree-p existence criterion. If a primitive cover of degree p^a exists, its overlattice contains an index-p intermediate overlattice, obtained by choosing an order-p subgroup of Λ'/Λ. Primitivity in the larger lattice implies primitivity in the intermediate one. Hence existence in any positive p-power degree is equivalent to degree-p existence.

Let r be the rational rank of all characters. A maximal independent set gives a nonempty intersection of codimension r, and no intersection has larger codimension. Thus $r = \operatorname{rank} L(\mathcal{B})$. For essential arrangements this equals d, so the poset supplies the remaining dimension in (1.4). □

3. The dimension qualification and all finite-field ranks

Theorem 1.2 answers the question in [1, Remark 3.2.7] affirmatively at a fixed ambient dimension, and intrinsically for essential arrangements. No assumption that the total intersection is a singleton is required.

Example 3.1 (The undecorated nonessential boundary)**.** The divisor $\{1\}$ in $\mathbb{C}^*$ and the divisor $\{x = 1\}$ in $(\mathbb{C}^*)^2$ both have a two-element layer poset. For the first, $d = 1$ and $k_p = 1$, so there is no primitive p-cover. For the second, $d = 2$ and $k_p = 1$, so there are p primitive covers of degree p. For example $(u, v) \mapsto (u, v^p)$ preserves connectedness of $\{u = 1\}$. Thus allowing the ambient dimension to vary gives a negative answer to the undecorated-poset assertion.

Corollary 3.2. *If the rational character rank is smaller than d, every prime admits a primitive p-cover. If $d \geq 2$, the primes without a primitive p-cover form a finite set, all satisfying $L_d(p) \leq n$. For a nonempty arrangement in dimension one, no prime admits one.*

Proof. If the rational rank is $r < d$, all $(r + 1)$-minors vanish integrally, so the reduced characters span a subspace of dimension at most r. Their projective lines cannot exhaust those of $\mathbb{F}_p^d$. In all cases $k_p \leq n$. Since $L_d(p)$ tends to infinity with p when $d \geq 2$, Theorem 1.2 gives the asserted finite restriction. When $d = 1$ and $n \geq 1$, every reduced character spans the unique line. □

The large-prime argument in [1, Corollary 3.2.8] applies for $d \geq 2$, where $L_d(p)$ tends to infinity. In dimension one, $L_1(p) = 1$ for every prime.

For the higher counts, it is convenient to extract independent-set indices from the whole poset. If $I \subseteq [n]$ is nonempty, let $\mathcal{U}(I)$ be the set of minimal common upper bounds of its atoms. These are minimal with respect to the reverse-inclusion order.

Lemma 3.3. *The characters indexed by I are rationally independent if and only if $\mathcal{U}(I)$ is nonempty and its elements have rank $|I|$. For independent I, put $m(I) = |\mathcal{U}(I)|$, and set $m(\varnothing) = 1$. Then*

$$r_p(S) = \max\{|I| : I \subseteq S,\ I \text{ rationally independent},\ p \nmid m(I)\}. \tag{3.1}$$

The maximum includes the empty set. In particular $L(\mathcal{B})$ determines every $r_p(S)$.

Proof. The intersection of the divisors indexed by I, if nonempty, is a finite union of cosets of the kernel of their character map. Each component has codimension equal to the rational row rank. Each such component is a layer and a minimal common upper bound; conversely every minimal common upper bound is one of these components. For independent I, the character map is surjective by Smith normal form, so the intersection is nonempty for arbitrary translation constants. This proves the independent-set test.

For an independent set of size k, the same Smith calculation shows that $m(I)$ is the product of its k Smith factors, hence the gcd of its maximal minors. Its reduction has rank k precisely when $p \nmid m(I)$. Every mod-p independent subset is rationally independent: a nonzero maximal minor modulo p is a nonzero integer minor. A basis of the reduced vectors in S therefore attains the right

side of (3.1), and every set counted on that right side is mod-p independent. This proves both inequalities. □

Minimal common upper bounds record the connected components of the intersection itself. Further intersections with other divisors may have larger codimension, even when the characters indexed by I are dependent; this explains the minimality condition in the lemma.

4. Every finite abelian deck-group type

For integers $0 \le b \le a$, write

$$\begin{bmatrix} a \\ b \end{bmatrix}_p = \prod_{j=0}^{b-1} \frac{p^{a-j}-1}{p^{b-j}-1}.$$

It is the number of b-dimensional subspaces of $\mathbb{F}_p^a$: count ordered independent b-tuples and divide by the number of ordered bases of $\mathbb{F}_p^b$. The value is one at $b=0$ and zero if $b \notin \{0,\dots,a\}$.

Proposition 4.1 (Elementary and cyclic types)**.** *For $0 \le s \le d$, the number of primitive covers with deck group $(\mathbb{Z}/p\mathbb{Z})^s$ is*

$$N_s(\mathcal{B},p) = \sum_{S\subseteq[n]} (-1)^{|S|} \begin{bmatrix} d-r_p(S) \\ s-r_p(S) \end{bmatrix}_p. \tag{4.1}$$

For $a \ge 1$, the number with cyclic deck group $\mathbb{Z}/p^a\mathbb{Z}$ is

$$p^{(a-1)(d-1)}\bigl(L_d(p) - k_p(\mathcal{B})\bigr). \tag{4.2}$$

Proof. An elementary overlattice corresponds to an s-dimensional subspace

$$W = \Lambda'/\Lambda \subseteq p^{-1}\Lambda/\Lambda \cong \mathbb{F}_p^d.$$

Divisibility of χ_i in the overlattice is equivalent to $\mathbb{F}_p\overline{\chi}_i \subseteq W$. Requiring this for every $i \in S$ means that W contains their span, of dimension $r_p(S)$. The number is the Gaussian coefficient in (4.1), by passing to the quotient by that span. Inclusion–exclusion gives the formula, including when some reduced lines coincide.

A cyclic overlattice of index p^a has the form $\Lambda + \mathbb{Z}(v/p^a)$, with v primitive modulo p^a. Two such generators give the same overlattice precisely when they differ by a unit modulo p^a. Its order-p subgroup is generated by v/p, so primitivity depends only on $\mathbb{F}_p\overline{v}$. For a fixed nonzero projective line, there are $(p-1)p^{(a-1)d}$ lifts of its nonzero vectors modulo p^a, and $(p-1)p^{a-1}$ unit multiples per overlattice. Their quotient is $p^{(a-1)(d-1)}$. There are $L_d(p) - k_p$ allowed lines, giving (4.2). □

Let $\lambda = (a_1 \ge \cdots \ge a_s > 0)$ be a partition and put

$$A_\lambda = \bigoplus_{j=1}^{s} \mathbb{Z}/p^{a_j}\mathbb{Z}, \qquad r_i = \#\{j : a_j \ge i\}.$$

Thus $r_1 = s$ and $r_i = 0$ for large i. Define

$$B_d(\lambda) = p^{\sum_{i\ge1} r_{i+1}(d-r_i)} \prod_{i\ge1} \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}_p \quad (s \le d). \tag{4.3}$$

For the empty partition the product is one. If $s > d$, no such finite-index lattice quotient is possible.

Theorem 4.2. *The number of primitive covers with deck group isomorphic to A_λ, for $s \le d$, is*

$$N_s(\mathcal{B},p)B_d(\lambda). \tag{4.4}$$

For a finite abelian group $A = \bigoplus_p A_{\lambda(p)}$, the number is the product of these expressions over its nontrivial primary parts. Consequently d and $L(\mathcal{B})$ determine the counts in every abelian deck-group type.

Proof. For a final p-primary overlattice Λ', use its canonical filtration

$$\Lambda_0 = \Lambda, \qquad \Lambda_i = \Lambda' \cap p^{-i}\Lambda. \tag{4.5}$$

If Λ'/Λ has type λ, then $\Lambda_i/\Lambda_{i-1} \cong \mathbb{F}_p^{r_i}$ and $p\Lambda_i \subseteq \Lambda_{i-1}$. The first lattice Λ_1 is an elementary overlattice of rank s. Every possible loss of primitivity is already detected there, since $\chi_j/p \in p^{-1}\Lambda$ for every $1 \le j \le n$. There are $N_s(\mathcal{B}, p)$ permissible choices for this first lattice.

Fix consecutive lattices $\Lambda_{i-1} \subseteq \Lambda_i$ already constructed canonically. The next candidate has

$$\Lambda_i \subseteq \Lambda_{i+1} \subseteq p^{-1}\Lambda_i$$

and is equivalent to choosing a subspace

$$W_{i+1} = p\Lambda_{i+1}/p\Lambda_i \subseteq \Lambda_i/p\Lambda_i, \qquad \dim W_{i+1} = r_{i+1}.$$

The condition that the filtration remain canonical is

$$\Lambda_{i+1} \cap p^{-i}\Lambda = \Lambda_i. \tag{4.6}$$

For $x \in \Lambda_{i+1}$, one has $px \in \Lambda_i$, and

$$x \in p^{-i}\Lambda \quad \Longleftrightarrow \quad px \in \Lambda_i \cap p^{-(i-1)}\Lambda = \Lambda_{i-1}.$$

Thus (4.6) holds exactly when the quotient map

$$\Lambda_i/p\Lambda_i \longrightarrow \Lambda_i/\Lambda_{i-1}$$

is injective on W_{i+1}. Indeed, multiplication by p identifies Λ_{i+1}/Λ_i with W_{i+1}, and the displayed equivalence identifies the kernel of the restricted map with $(\Lambda_{i+1} \cap p^{-i}\Lambda)/\Lambda_i$. The quotient Λ_i/Λ_{i-1} has dimension r_i, and the kernel of the quotient map has dimension $d - r_i$.

Choose the r_{i+1}-dimensional image in that target in $\begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}_p$ ways. Its lifts disjoint from the kernel are graphs of linear maps to a $(d - r_i)$-dimensional space, giving $p^{r_{i+1}(d-r_i)}$ choices. Each graph determines one next lattice. Multiplying these independent counts over i gives (4.3).

A final lattice determines (4.5) uniquely. Conversely, suppose the construction has reached Λ_{i+1}. For $0 \le j \le i$, the inclusion $p^{-j}\Lambda \subseteq p^{-i}\Lambda$ and (4.6) give

$$\Lambda_{i+1} \cap p^{-j}\Lambda = \Lambda_i \cap p^{-j}\Lambda = \Lambda_j.$$

Induction therefore preserves all earlier intersections. At the endpoint, the resulting lattice has a canonical filtration with the prescribed r_i. Since $\Lambda_i/\Lambda = (\Lambda'/\Lambda)[p^i]$, the order of $(\Lambda'/\Lambda)[p^i]$ is then $p^{r_1+\cdots+r_i}$, so the finite abelian group has exactly type λ. The construction thus counts each lattice of type λ once. All primitivity conditions depend on Λ_1, establishing (4.4).

For a general finite quotient, its primary decomposition gives the unique lattices

$$\Lambda'_p = \{x \in \Lambda' : x + \Lambda \text{ belongs to the } p\text{-primary part}\}.$$

Conversely the sum of these primary overlattices is the unique final overlattice. Primitivity is checked prime by prime, hence the counts multiply. Finally the deck group is the finite abelian dual of the lattice quotient, which has the same isomorphism type. Each overlattice specifies one covering isomorphism class. □

5. The local series and its characteristic-polynomial numerator

Write $g_m(d)$ for the number of all connected ambient covers of degree m, without divisor conditions. Their local generating series has the following form.

Lemma 5.1. *For $d \ge 1$,*

$$G_{d,p}(z) := \sum_{k \ge 0} g_{p^k}(d) z^k = \prod_{j=0}^{d-1} (1 - p^j z)^{-1}. \tag{5.1}$$

As arithmetic functions,

$$g(d) = \mathrm{id}^0 * \mathrm{id}^1 * \cdots * \mathrm{id}^{d-1}, \qquad \mathrm{id}^j(m) = m^j, \tag{5.2}$$

where $*$ is Dirichlet convolution.

Proof. Duality replaces overlattices by sublattices of $\mathbb{Z}^d$ without changing indices. Use the unique lower triangular column Hermite form with positive diagonal $h_1, \dots, h_d$ and, in row i, its $i-1$ earlier entries reduced to residues modulo h_i. A fixed diagonal gives $\prod_i h_i^{i-1}$ sublattices, of index $\prod_i h_i$. This normal form follows successively by taking the positive generator of the first-coordinate image, then doing the same in its coordinate kernel, and reducing the remaining coefficients by the later diagonal columns. Those residues determine and distinguish all resulting sublattices. Summing over all diagonal tuples of product m proves (5.2). For $h_i = p^{u_i}$, summing independently over $u_i \geq 0$ gives the geometric series product (5.1). □

Proposition 5.2. *Equation (1.7) holds. If M_p is the vector matroid of the reduced characters and $r_p = r_p([n])$, its ordinary characteristic polynomial satisfies*

$$P_{\mathcal{B},p}(z) = z^{r_p} \chi_{M_p}(z^{-1}). \tag{5.3}$$

Proof. Within p-primary overlattices, divisibility of χ_i is equivalent to $\chi_i/p \in \Lambda'$. Simultaneous divisibility for all $i \in S$ is equivalent to containment of

$$\Delta_S = \Lambda + \sum_{i \in S} \mathbb{Z}(\chi_i/p).$$

This is a rank-d lattice with $[\Delta_S : \Lambda] = p^{r_p(S)}$. Every larger finite p-primary overlattice of Δ_S is counted by $G_{d,p}$, and its index relative to Λ is multiplied by that fixed power of p. Thus the generating function for this intersection event is $z^{r_p(S)} G_{d,p}(z)$. Finite inclusion–exclusion over the labels proves (1.7). Coincident reduced lines simply give repeated events to which the same formula still applies.

The characteristic polynomial has the Whitney subset expression

$$\chi_{M_p}(u) = \sum_{S \subseteq [n]} (-1)^{|S|} u^{r_p - r_p(S)}.$$

Substitution proves (5.3); this expression may also be taken as the definition of the characteristic polynomial here. □

Every character has nonzero reduction, so no nonempty set has rank zero. Hence the constant coefficient of $P_{\mathcal{B},p}$ is one, its degree is at most d, and the sum of the absolute values of its coefficients is at most 2^n. Except at primes dividing one of the finitely many nonzero independent indices $m(I)$, all $r_p(S)$ agree with their rational ranks by Lemma 3.3. Thus the numerator is one fixed polynomial at every remaining prime.

6. Global counting and positive density

Let

$$B_{\mathcal{B}}(s) = \prod_p P_{\mathcal{B},p}(p^{-s}).$$

To justify its use as a Dirichlet series, write $P_{\mathcal{B},p}(z) = 1 + \sum_{j=1}^{d} c_{p,j} z^j$ and define the multiplicative coefficients b_m by $b_1 = 1$, $b_{p^j} = c_{p,j}$ for $1 \leq j \leq d$, and $b_{p^j} = 0$ for $j > d$. For every real $\sigma > 1$,

$$\sum_{m \geq 1} \frac{|b_m|}{m^\sigma} = \prod_p \left(1 + \sum_{j=1}^{d} |c_{p,j}| p^{-j\sigma} \right) \leq \prod_p (1 + 2^n p^{-\sigma}) < \infty. \tag{6.1}$$

Indeed the last product converges because $\sum_p p^{-\sigma} \leq \sum_{m \geq 2} m^{-\sigma} < \infty$. The equality follows first for finitely many primes, then by monotone convergence of the nonnegative coefficient sums.

Consequently the Euler product for $B_{\mathcal{B}}(s)$ represents an absolutely convergent Dirichlet series in $\operatorname{Re} s > 1$.

The primary lattice decomposition in the proof of Theorem 4.2 makes a_m multiplicative. Together with Proposition 5.2 and Lemma 5.1, the local coefficient identities give

$$a = g(d) * b. \tag{6.2}$$

This equality also holds formally, because each coefficient involves finitely many divisors. In $\operatorname{Re} s > d$, all series involved are absolutely convergent: $a_m \le g_m(d)$, and (5.2) gives

$$\sum_{m\ge 1} \frac{g_m(d)}{m^s} = \prod_{j=0}^{d-1} \zeta(s-j).$$

This proves (1.8).

To prove the asymptotic, assume $d \ge 2$ and put

$$h = \mathrm{id}^0 * \mathrm{id}^1 * \cdots * \mathrm{id}^{d-2}, \qquad C_d = \frac{\prod_{j=2}^{d} \zeta(j)}{d}.$$

All coefficients of h are nonnegative, and

$$\sum_{\ell\ge 1} \frac{h_\ell}{\ell^d} = \prod_{j=2}^{d} \zeta(j) < \infty. \tag{6.3}$$

Writing $G(X) = \sum_{m\le X} g_m(d)$ and taking $G(X) = 0$ for $X < 1$, the convolution gives

$$G(X) = \sum_{\ell\le X} h_\ell \sum_{k\le X/\ell} k^{d-1}.$$

For a fixed ℓ, integral comparison for the power sum gives

$$X^{-d} \sum_{k\le X/\ell} k^{d-1} \longrightarrow \frac{1}{d\ell^d}.$$

For all $X \ge 1$, interpreting the term as zero for $\ell > X$, its absolute value is bounded by C/ℓ^d, with C depending only on d: the power sum is at most CY^d for $Y \ge 1$. Equation (6.3) supplies a summable dominating sequence. Hence

$$G(X) \sim C_d X^d, \qquad G(X) \le C' X^d \quad (X \ge 1). \tag{6.4}$$

Now use (6.2):

$$X^{-d} \sum_{m\le X} a_m = \sum_{\ell\le X} b_\ell X^{-d} G(X/\ell).$$

For each fixed ℓ, the summand tends to $C_d b_\ell/\ell^d$; for all X its absolute value is at most $C'|b_\ell|/\ell^d$. Since $d \ge 2$, (6.1) at $\sigma = d$ proves summability. Dominated convergence therefore gives

$$X^{-d} \sum_{m\le X} a_m \longrightarrow C_d B_{\mathcal{B}}(d). \tag{6.5}$$

It remains to show that this limit is strictly positive. The series (1.7) converges at $z = p^{-d}$: its coefficients are nonnegative and at most those of $G_{d,p}$, whose radius is $p^{-(d-1)}$. The left side is at least one. All factors $1 - p^{j-d}$ in its denominator are positive. It follows that

$$0 < P_{\mathcal{B},p}(p^{-d}) \le 1, \tag{6.6}$$

where the upper bound again follows by comparison with $G_{d,p}$. Also

$$|P_{\mathcal{B},p}(p^{-d}) - 1| \le 2^n p^{-d}.$$

For $d \ge 2$ the sum of these deviations is finite. The logarithms of all sufficiently large positive factors are thus absolutely summable, and the finitely many remaining factors are positive by (6.6). The

product is strictly positive and equals $B_{\mathcal{B}}(d) = D_{\mathcal{B}}$. Equations (6.5) and (6.4) prove the asymptotic and density in Theorem 1.3.

At a prime for which no positive-degree primitive p-primary cover exists, the local series is one. Its correction factor in (1.9) is then $\prod_{j=0}^{d-1}(1-p^{j-d})$, still strictly positive. Such primes therefore contribute positive factors to $D_{\mathcal{B}}$.

7. Examples

Example 7.1 (Parallel translated divisors)**.** In dimension two, let all divisors have the same primitive direction, with distinct constants. For each prime their reduced lines coincide, so $k_p = 1$ and $P_{\mathcal{B},p}(z) = 1 - z$. Hence

$$\sum_{k\geq 0} a_{p^k} z^k = \frac{1}{1-pz}.$$

There are p^k primitive covers of degree p^k, and the same number of cyclic covers of that degree. An elementary cover of rank two would contain every line and is forbidden, so all permissible primary types are cyclic in this example. Globally $a_m = m$, so $\sum_{m\leq X} a_m \sim X^2/2$. The density formula gives $D_{\mathcal{B}} = \prod_p (1-p^{-2}) = \zeta(2)^{-1}$, agreeing with Theorem 1.3.

Example 7.2 (A congruent pair of directions)**.** Take $\chi_1 = (1,0)$ and $\chi_2 = (1,m)$ in dimension two, with $m \neq 0$. For any nonzero constants, the two divisors meet in $|m|$ points. If $p \mid m$, their reduced lines coincide and

$$\sum_{k\geq 0} a_{p^k} z^k = \frac{1}{1-pz}.$$

If $p \nmid m$, their lines are distinct and

$$P_{\mathcal{B},p}(z) = (1-z)^2, \qquad \sum_{k\geq 0} a_{p^k} z^k = \frac{1-z}{1-pz}.$$

The positive-degree coefficients in the second case are $(p-1)p^{k-1}$. Thus the two local series are distinguished by the divisibility of the intersection count $|m|$. For $|m| = 1$, the count in every degree r is Euler's totient $\varphi(r)$, and $D_{\mathcal{B}} = \zeta(2)^{-2}$. The asymptotic is consequently $\sum_{r\leq X} \varphi(r) \sim X^2/(2\zeta(2))$.

Example 7.3 (An excluded primary direction)**.** For $\chi_1 = (1,0)$, $\chi_2 = (0,1)$, $\chi_3 = (1,-1)$, every pair is independent over every field, and the triple has rank two. Thus

$$P_{\mathcal{B},p}(z) = 1 - 3z + 2z^2 = (1-z)(1-2z).$$

At $p = 2$ this cancels the whole denominator in (1.7), giving local series one: there is no nontrivial primitive 2-primary ambient cover of any deck-group type. At odd primes the projective lines leave room for primitive covers.

Example 7.4 (Dimension zero, dimension one, and the empty arrangement)**.** If $d = 0$, the only arrangement is empty and the only connected cover has degree one. If $d = 1$ and the arrangement is nonempty, its divisors are points. Every nontrivial ambient cover splits the inverse image of each point, so $a_1 = 1$ and $a_m = 0$ for $m > 1$. Here $P_{\mathcal{B},p}(z) = 1 - z$ cancels the sole denominator in (1.7). The cover density is zero; it falls outside the $d \geq 2$ assertion. For an empty arrangement in any positive dimension, $P_{\mathcal{B},p} = 1$, all covers are permissible, and the density is one.

The prime-degree formula uses only rank-two layers. The counts for all abelian deck-group types and the generating functions use the independent intersection indices $m(I)$ from the full layer poset. These indices determine the reduced character ranks through Lemma 3.3, so all the formulas depend only on the layer poset and the ambient dimension.

References

[1] E. Delucchi and E. Marmo, *Toric arrangements and Bloch–Kato pro-p groups*, preprint, 2025. arXiv:2507.16428v1. The question and result numbers cited here refer to this version.
[2] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
[3] R. Pagaria, *Combinatorics of toric arrangements*, Rend. Lincei Mat. Appl. **30** (2019), 317–349. arXiv:1710.00409.
[4] R. Pagaria and G. Paolini, *Representations of torsion-free arithmetic matroids*, European J. Combin. **93** (2021), 103272. doi:10.1016/j.ejc.2020.103272.